
MATH 135 Fall 2020: Extra Practice 3
Warm-Up Exercises**WE01.** Prove the following two quantified statements.

(a) $\forall n \in \mathbb{N}, n + 1 \geq 2$

Proof. Let $n \in \mathbb{N}$. Recall that 1 is the smallest natural. $n \geq 1 \iff n + 1 \geq 2$. \square

(b) $\exists n \in \mathbb{Z}, \frac{5n-6}{3} \in \mathbb{Z}$

Proof. Select $n = 3$. Then, $\frac{5n-6}{3} = \frac{15-6}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$. \square **WE02.** Prove that for all $k \in \mathbb{Z}$, if k is odd, then $4k + 7$ is odd.*Proof.* Let k be an odd integer. Then, it can be written as $2n + 1$ for some integer n .Substituting, $4k + 7 = 4(2n + 1) + 7 = 8n + 11 = 2(4n + 5) + 1$. By definition, since $4k + 7$ can be written as $2m + 1$ where $m = 4n + 5$ is an integer, it is odd. \square **WE03.** Consider the following proposition*For all $a, b \in \mathbb{Z}$, if $a^3 \mid b^3$, then $a \mid b$.*

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

(a) *Consider $a = 2, b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.*

This proof is erroneous as it only considers one specific case of a and b and not the general case of integer a and b .

(b) *Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3 a^3$. Since $k^3 \in \mathbb{Z}$, then $a^3 \mid b^3$.*

This proof supposes the conclusion instead of the hypothesis.

(c) *Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.*

The proof does not guarantee that $\frac{ka^2}{b^2}$ is an integer.**WE04.** Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$, then $x \neq 5$.*Proof.* Suppose for the contrapositive that $x = 5$. Then, $x^3 - 5x^2 + 3x = (5)^3 - 5(5)^2 + 3(5) = 15$, as required. Since the contrapositive is true, the original implication must be true. \square **WE05.** Prove that there do not exist integers x and y such that $2x + 4y = 3$.

Proof. For the sake of contradiction, suppose the negation is true.

Consider the negation of the statement: there exist integers x and y such that $2x + 4y = 3$. Let x and y be such integers. Then, $x + 2y$ is an integer. Therefore, $2x + 4y = 2(x + 2y)$ is even. However, 3 is odd. An integer cannot be both even and odd, therefore, the negation is false, and the original statement is true. \square

WE06. Prove that an integer is even if and only if its square is an even integer.

Proof. (\Rightarrow) Let n be an even integer. Then, $n = 2k$ for some integer k . $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

(\Leftarrow) Let n be an even square integer. Then, $n = 2k$ for some integer k and $n = x \cdot x$ for some integer x . Since $2k = x \cdot x$, and 2 is prime, 2 must divide x . Therefore, $x = 2y$ for some integer y , which is the definition of being even.

Since the implication is true in both directions, the biconditional is true. \square

Recommended Problems

RP01. Prove that $x^2 + 9 \geq 6x$ for all real numbers x .

Proof. Let x be a real number. $x^2 + 9 \geq 6x \iff x^2 - 6x + 9 \geq 0 \iff (x - 3)^2 \geq 0$. Since the square of a real is always non-negative, the statements are true. \square

RP02. Prove that for all $r \in \mathbb{R}$ where $r \neq -1$ and $r \neq -2$,

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r(2^r)}{(r+1)(r+2)}$$

Proof. Let r be a real number that is neither -1 nor -2 . Then,

$$\begin{aligned} LHS &= \frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} \\ &= \frac{2^{r+1}(r+1) - 2^r(r+2)}{(r+1)(r+2)} \\ &= \frac{r2^{r+1} + 2^{r+1} - r2^r - 2 \cdot 2^r}{(r+1)(r+2)} \\ &= \frac{r2^{r+1} + 2^{r+1} - r2^r - 2^{r+1}}{(r+1)(r+2)} \\ &= \frac{r(2^{r+1} - 2^r)}{(r+1)(r+2)} \\ &= \frac{r(2^r \cdot 2 - 2^r)}{(r+1)(r+2)} \\ &= \frac{r(2^r + 2^r - 2^r)}{(r+1)(r+2)} \\ &= \frac{r(2^r)}{(r+1)(r+2)} \\ &= RHS \end{aligned}$$

Since the left side equals the right side, the equality is true. \square

RP03. Prove that there exists a real number x such that $x^2 - 6x + 11 \leq 2$.

Proof. Let $x = 3$. $x^2 - 6x + 11 = (3)^2 - 6(3) + 11 = 9 - 18 + 11 = 2 \leq 2$, as required. Since 3 is a real number, the statement is true. \square

RP04. Prove or disprove each of the following statements.

(a) $\forall n \in \mathbb{Z}, \frac{5n-6}{3}$ is an integer.

Proof. Let $n = 1$ as a counter-example. Then, $\frac{5n-6}{3} = \frac{5-6}{3} = -\frac{1}{3}$, which is not an integer. Therefore, the statement is false. \square

(b) $\forall a \in \mathbb{Z}, a^3 + a + 2$ is even.

Proof. Let a be an integer. Then, a is either even or odd. Suppose that a is even and can be written as $a = 2k$ for an integer k . Then, $a^3 + a + 2 = (2k)^3 + 2k + 2 = 8k^3 + 2k + 2 = 2(4k^3 + k + 1)$, an even number.

Suppose a is odd and can be written as $a = 2k + 1$ for an integer k . Then, $a^3 + a + 2 = (2k + 1)^3 + (2k + 1) + 2 = 8k^3 + 12k^2 + 8k + 4 = 2(4k^3 + 6k^2 + 4k + 2)$, an even number.

Therefore, the statement is true. \square

(c) For every prime number p , $p + 7$ is composite.

Proof. Let p be a prime number.

If p is even, then $p = 2$, and $p + 7 = 9$ which is composite.

If p is odd, $p = 2k + 1$ for some integer $k \geq 0$ (as there are no negative primes). Then, $p + 7 = 2k + 8 = 2(k + 4)$, which is even. The only even prime is 2, but $2k + 8 \geq 8$, so $p + 7$ is composite.

Therefore, since all primes are either even or odd, $p + 7$ is composite for all primes. \square

(d) For all $x \in \mathbb{R}, |x - 3| + |x - 7| \geq 10$.

Proof. Let $x = 3$ as a counter-example. Then, $|x - 3| + |x - 7| = |(3) - 3| + |(3) - 7| = 0 + 4 = 4 \not\geq 10$. Therefore, the statement is false. \square

(e) There exists a natural number $m < 123456$ such that $123456m$ is a perfect square.

Proof. Let $m = 1929$, which is a natural number less than 123456. Then, $123456m = 238146624 = 15432^2$. Since $123456m$ can be written as n^2 where $n = 15432 \in \mathbb{Z}$, it is a perfect square, and the statement is true. \square

(f) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

Proof. Consider the negation, $\forall k \in \mathbb{Z}, 8 \mid (4k^2 + 12k + 8)$. Notice that the open sentence is logically equivalent to $8 \mid (4k^2 + 12k)$. Let k be a natural number. Then, k is either even or odd.

Suppose that k is even and can be written as $k = 2n$. Then, $4k^2 = 16n^2 = 8(2n^2)$, so $8 \mid 4k^2$. Likewise, $12k = 24n = 8(3n)$, so $8 \mid 12k$. By DIC, $8 \mid (4k^2 + 12k)$.

Now, suppose that k is odd and can be written as $k = 2n + 1$. Then, $4k^2 + 12k = 4(4n^2 + 2n + 1) + 12(2n + 1) = 16n^2 + 40n + 16 = 8(2n^2 + 5n + 1)$, so $8 \mid (4k^2 + 12k)$.

Therefore, the negation is true, so the original statement is false. \square

RP05. Prove or disprove each of the following statements involving nested quantifiers.

- (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.

Proof. Let n be an integer. If $n = 0$ or $n = \pm 1$, $n^3 - n = 0$ and all integers (including any k) divide zero.

If $n > 1$, we select $k = n + 1 > 2$. Factor: $n^3 - n = n(n - 1)(n + 1)$. Then, $n^3 - n = [n(n - 1)](n + 1)$, so $k \mid (n^3 - n)$.

If $n < 1$, first let $m = -n$ so $n^3 - n = (-m)^3 + m = -(m^3 - m)$. Now, select $k = m + 1 > 2$. Then, $n^3 - n = -m(m - 1)(m + 1)$, so $k \mid (n^3 - n)$.

Therefore, the statement is true. \square

- (b) For every positive integer a , there exists an integer b with $|b| < a$ such that b divides a .

Proof. We disprove by counter-example. Let $a = 1$. Then, $|b| < 1$, and the only such integer is 0. However, $0 \nmid 1$ since there is no integer k where $k \cdot 0 = 1$. Therefore, the statement is false. \square

- (c) There exists an integer n such that $m(n - 3) < 1$ for every integer m .

Proof. Choose $n = 3$ and let m be an integer. Then, $m(n - 3) = m(3 - 3) = 0 < 1$, as desired. Therefore, the statement is true. \square

- (d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$

Proof. Consider the negation $\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, -nm \geq 0$. Let n be a natural number.

We can choose an integer m , namely $m = -1$. Notice that because n is a natural number, $n > 0 \iff n(-1)(-1) > 0 \iff -nm > 0 \iff -nm \geq 0$.

Because the negation is true, the original statement is false. \square

RP06. Prove that for all integers a and b , if $a \mid (2b + 3)$ and $a \mid (3b + 5)$, then $a \mid 13$.

Proof. Let a and b be arbitrary integers, and assume that $a \mid (2b + 3)$ and $a \mid (3b + 5)$.

Recall the divisibility of integer combinations: since $2b + 3$ and $3b + 5$ are integers, a must divide $n(2b + 3) + m(3b + 5)$ for all integers n and m . Specifically, let $n = -39$ and $m = 26$. Then, $n(2b + 3) + m(3b + 5) = -78b - 117 + 78b + 130 = 13$. Therefore, $a \mid 13$. \square

RP07. Let a, b, c and d be positive integers. Prove that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof. Let a, b, c and d all be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$, which means $ad < bc$, because b and d are positive. Now, adding ab and cd to both sides, respectively:

$$\begin{array}{ll} ad < bc & ad < bc \\ ad + ab < bc + ab & ad + cd < bc + cd \\ a(b+d) < b(c+a) & d(a+c) < c(b+d) \\ \frac{a}{b} < \frac{a+c}{b+d} & \frac{a+c}{b+d} < \frac{c}{d} \end{array}$$

Therefore, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

RP08. Prove that for all integers n , if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

Proof. Let n be an integer where $1 - n^2 > 0$. Since squares of integers are positive, $1 > n^2$. This is only true when $|n| < 1$, but the only such integer is 0. $3(0) - 2 = -2$, which is even. □

RP09. Let a and b be integers. Prove each of the following implications.

(a) If $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$

Proof. Let a and b be integers with product 4.

Consider the possible values for a and b . 4's divisor pairs are $(\pm 1, \pm 4)$ and $(\pm 2, \pm 2)$. For all of these pairs, either $a = b$ or $a = b \pm 3$. Specifically:

- If $b = \pm 2$, then $a = b$
- If $b = 1$, then $a = 4 = b + 3$ (for $b = -1$, $a = -4 = b - 3$)
- If $b = 4$, then $a = 1 = b - 3$ (for $b = -4$, $a = -1 = b + 3$)

Notice that the conclusion factors to $(a - b)(a - b - 3)(a - b + 3) = 0$. This is true when $a = b$ or $a = b \pm 3$, which we just showed. □

(b) If a and b are positive, then $a^2(b + 1) + b^2(a + 1) \geq 4ab$

Proof. Let a and b be positive integers, i.e., at least 1.

If a and b are both at least 1, then $a + b \geq 2$, or $a + b - 2 \geq 0$. Likewise, ab is a positive integer, so $ab(a + b - 2) \geq 0$.

$$\begin{aligned} ab(a + b - 2) &\geq 0 \\ a^2b + b^2a - 2ab &\geq 0 \end{aligned}$$

Recall that squares are non-negative:

$$\begin{aligned} (a - b)^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 - 2ab + b^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 + a^2b + b^2 + b^2a &\geq 4ab \\ a^2(b + 1) + b^2(a + 1) &\geq 4ab \end{aligned}$$

□

RP10. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

Proof. Let a, b, c , and d be integers where $a \mid b$, $b \mid c$, and $c \mid d$.

Recall the transitivity of divisibility: for integers x, y , and z , if $x \mid y$ and $y \mid z$, then $x \mid z$.

Then, $a \mid b$ and $b \mid c$ implies $a \mid c$. Likewise, $a \mid c$ and $c \mid d$ implies $a \mid d$. \square

RP11. Prove that the product of any four consecutive integers is one less than a perfect square.

Proof. The statement is equivalently expressed that for any integer k , $k(k+1)(k+2)(k+3) = r^2 - 1$ for some positive integer r .

Let k be an integer. The product $k(k+1)(k+2)(k+3)$ expands to $k^4 + 6k^3 + 11k^2 + 6k$. As a fourth-degree polynomial, its square root would be a quadratic.

Expanding algebraically, the square of a quadratic in x , $ax^2 + bx + c$, is $a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2$.

Notice that when $a = c = 1$ and $b = 3$, this formula becomes $x^4 + 6x^3 + 11x^2 + 6x + 1$. The coefficients on x are precisely our original product (with a constant $+1$). Therefore, $x^4 + 6x^3 + 11x^2 + 6x = (x^2 + 3x + 1)^2 - 1$ for all real x .

We can now let $r = k^2 + 3k + 1$, which is a positive integer such that

$$\begin{aligned} r^2 - 1 &= (k^2 + 3k + 1)^2 - 1 \\ &= k^4 + 6k^3 + 11k^2 + 6k + 1 - 1 \\ &= k(k+1)(k+2)(k+3) \end{aligned}$$

and conclude that the statement is true. \square

RP12. Let $x, y \in \mathbb{R}$. Prove that if $xy + 2x - 3y - 6 < 0$, then $x < 3$ or $y < -2$.

Proof. Let x and y be real solutions to $xy + 2x - 3y - 6 < 0$.

Notice that the inequality factors to $(x - 3)(y + 2) < 0$. This is true when x and y are non-zero and have opposite signs: either $x < 3$ and $y > -2$, or $x > 3$ and $y < -2$. Therefore, either $x < 3$ or $y < -2$. \square

RP13. Is the following implication true for all integers a, b and c ? Prove that your answer is correct.

$$a \mid b \text{ if and only if } ac \mid bc$$

Solution. The statement is false. Consider the counterexample $a = 2$, $b = 3$, and $c = 0$. Then, the backwards implication's hypothesis is true ($0 \mid 0$) but the conclusion is false ($2 \nmid 3$). \square

RP14. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

Proof. Consider the two implications of the biconditional statement:

(\Rightarrow) Let n be an integer where 2 divides $n^4 - 3$. This means there is an integer k where $n^4 - 3 = 2k$. Notice that this means $n^4 - 3$ is even, so $n^4 = 2(k + 1) + 1$ is odd. Even numbers raised to the fourth power remain even, so n must be odd. Therefore, $n = 2m + 1$ for some integer m .

Now, expand $n^2 + 3$:

$$\begin{aligned} n^2 + 3 &= (2m + 1)^2 + 3 \\ &= 4m^2 + 4m + 1 + 3 \\ &= 4(m^2 + m + 1) \end{aligned}$$

Because $m^2 + m + 1$ is an integer, $4 \mid (n^2 + 3)$.

(\Leftarrow) Let n be an integer where 4 divides $n^2 + 3$. This means there is an integer k where $n^2 + 3 = 4k$ or $n^2 = 4k - 3$, and

$$\begin{aligned} n^2 &= 4k - 3 \\ n^4 &= (4k - 3)^2 \\ n^4 &= 16k^2 - 24k + 9 \\ n^4 - 3 &= 16k^2 - 24k + 6 \\ &= 2(8k^2 - 12k + 3) \end{aligned}$$

Because $8k^2 - 12k + 3$ is an integer, $2 \mid (n^4 - 3)$.

Therefore, since both expressions imply the other, $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$. \square

RP15. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

Proof. Consider the contrapositive, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$.

Let x and y be non-zero integers. WLOG, take $x \leq y$.

Now, take cases of the signs of x and y :

- If $0 < x \leq y$, then $xy > 0$, since two positive numbers' product is a positive number.
- xy is also positive when $x \leq y < 0$, with two negative numbers.
- When $x < 0 < y$, i.e. the signs are opposite, $xy < 0$.

Since xy can never be 0 for any combination of non-zero integers, the contrapositive, and by extension, the original implication, is true. \square

RP16. Prove that $\forall a, b \in \mathbb{Z}, [(a \mid b \wedge b \mid a) \iff a = \pm b]$.

Proof. Let a and b be integers. Suppose a divides b and vice versa. Equivalently, integers p and q exist such that $a = pb$ and $b = qa$. Substituting, $a = pb = p(qa) \iff 1 = pq \iff p = \frac{1}{q}$.

The only integers of the form $\frac{1}{k}$ with integer k are 1 and -1. Therefore, $p = \frac{1}{q}$ if and only if $p = \pm 1$, i.e., $a = \pm b$. \square

RP17. Let a be an integer. Prove that $a^2 + 2a - 3$ is even if and only if a is odd.

Proof. Consider the two implications of the biconditional statement:

(\Rightarrow) Let a be an odd integer, or, $a = 2k + 1$ for some integer k . Then,

$$\begin{aligned} 2a^2 + 2a - 3 &= (2k + 1)^2 + 2(2k + 1) - 3 \\ &= 4k^2 + 4k + 1 + 4k + 2 - 3 \\ &= 4k^2 + 8k - 2 \\ &= 2(2k^2 + 4k - 1) \end{aligned}$$

which is even, because $2k^2 + 4k - 1$ is an integer.

(\Leftarrow) Consider the contrapositive, where even a implies odd $a^2 + 2a - 3$. Let a be an even integer, i.e., $a = 2k$ for some integer k . Then,

$$\begin{aligned} a^2 + 2a - 3 &= (2k)^2 + 2(2k) - 3 \\ &= 4k^2 - 4k - 3 \\ &= 2(2k^2 - 2k - 2) + 1 \end{aligned}$$

which is odd, because $2k^2 - 2k - 2$ is an integer. Since the contrapositive is true, the original implication is also true.

Therefore, since both implications hold, the statement is true. \square

RP18. Prove or disprove each of the following for any integers x and y .

(a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.

(b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.

Proof. First, notice that if $2 \mid n$ for an integer n , then $n = 2k$ for some integer k . This is the definition of saying n is even. Therefore, $2 \nmid n$ is the same as saying n is not even, i.e., n is odd.

Let x and y be odd integers. Equivalently, $x = 2p + 1$ and $y = 2q + 1$ for some integers p and q . Substituting into xy , $(2p + 1)(2q + 1) = 2pq + 2p + 2q + 1 = 2(pq + p + q) + 1$. By definition, since $pq + p + q$ is an integer, xy is odd.

Therefore, x and y are odd if and only if xy is odd, so (a) and (b) are both true. \square

(c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

Proof. Consider the contrapositive, “if $10 \mid x$ and $10 \mid y$ then $10 \mid xy$ ”. Let x and y be integers where 10 divides both.

This means $x = 10n$ and $y = 10m$ for some integers n and m . Then, $xy = (10n)(10m) = 10(10nm)$, and since $10nm$ is an integer, $10 \mid xy$.

Since the contrapositive, the original implication is true. \square

(d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

Proof. For a counterexample, let $x = 5$ and $y = 2$. $10 \nmid x$ and $10 \nmid y$ since $2 < 5 < 10$. However, $xy = 10$ and $10 \mid 10$, so the statement is false. \square

RP19. For every odd integer n , prove that there exists a unique integer m such that $n^2 = 8m + 1$.

Proof. Let n be an odd integer, i.e., $n = 2k + 1$ for some other integer k . Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. We must show that $8m = 4k^2 + 4k \iff 2m = k^2 + k$, or, $k^2 + k$ is even. Now, consider k 's parity:

Suppose k is even. Then, $k = 2p$ for an integer p and $k^2 + k = 4p^2 + 2p = 2(2p^2 + p)$, which means that $k^2 + k$ is even.

Now, suppose k is odd. Then, $k = 2p + 1$ for an integer p and $k^2 + k = 4p^2 + 6p + 2 = 2(2p^2 + 3p + 1)$, which means that $k^2 + k$ is even.

Since k is either even or odd, $k^2 + k$ is even for all k .

Therefore, $m = \frac{k^2 + k}{2}$ is an integer, but recall $k = \frac{n-1}{2}$, so:

$$m = \frac{k^2 + k}{2} = \frac{\left(\frac{n-1}{2}\right)^2 + \frac{n-1}{2}}{2} = \frac{\frac{(n-1)^2}{4} + \frac{n-1}{2}}{2} = \frac{(n-1)^2 + 2(n-1)}{8} = \frac{n^2 - 1}{8}$$

is the same integer, but $m = \frac{n^2 - 1}{8}$ if and only if $n^2 = 8m + 1$, so the statement is true. \square

RP20. Prove the following statements.

(a) There is no smallest positive real number.

Proof. Suppose, for a contradiction, that there is a smallest positive real number n . Now, consider $\frac{n}{2}$.

This number is still real (\mathbb{R} is closed under division). $\frac{n}{2}$ is positive because n and 2 are positive. Therefore, $\frac{n}{2}$ is a positive real number.

Clearly $\frac{n}{2} < n$, so the supposition must be false. Therefore, there is no smallest positive real number. \square

(b) For every even integer n , n cannot be expressed as the sum of three odd integers.

Proof. We will prove by the contrapositive. Let r , s , and t be arbitrary integers so $2r + 1$, $2s + 1$, and $2t + 1$ are odd.

Then, $r + s + t = 2r + 2s + 2t + 3 = 2(r + s + t + 1) + 1$, so this sum is odd.

Therefore, the sum of three odd integers is always odd, and no even integer may be expressed as such a sum. \square

(c) Let $a, b \in \mathbb{Z}$. If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.

Proof. Let a and b be integers and suppose, for a contradiction, that the negation is true. Then, a is even, b is odd, and $4 \mid (a^2 + 2b^2)$.

Rewrite $a = 2n$ and $b = 2m + 1$ with some integers n and m . Now, expand $a^2 + 2b^2 = (2n)^2 + 2(2m + 1)^2 = 4n^2 + 8m^2 + 8m + 2$.

We can extract a factor of four, and get $4 \mid (4(n^2 + 2m^2 + 2m) + 2)$. Then, 4 must divide 2, which is a contradiction.

Therefore, the negation is false, so the original statement is true. \square

- (d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2 + 3y^2 = m$.

Proof. Those negations are ugly so we can consider the contrapositive:

If $x^2 + 3y^2 = m$ has integer solutions in x and y , m is odd or $4 \mid m$.

Notice that $2 \nmid m \vee 4 \mid m \equiv 4 \mid m$.

Suppose integers x and y so $x^2 + 3y^2 = m$ exist. Break into cases for x and y 's parities.

- If x and y are odd, they can respectively be expressed as $2p + 1$ and $2q + 1$ for integers p and q . Then, $m = (2p + 1)^2 + 3(2q + 1)^2 = 4p^2 + 4p + 1 + 3(4q^2 + 4q + 1)$. This simplifies to $4(p^2 + 3q^2 + p + 3q + 1)$, so $m \mid 4$.
- If x and y are even, let $x = 2p$ and $y = 2q$. Then, $m = (2p)^2 + 3(2q)^2 = 4p^2 + 12q^2 = 4(p^2 + 3q^2)$, so $m \mid 4$.
- If x is odd and y is even, let $x = 2p + 1$ and $y = 2q$. Then, $m = (2p + 1)^2 + 3(2q)^2 = 4p^2 + 4p + 1 + 3(4q^2) = 2(2p^2 + 2p + 6q^2) + 1$, so m is odd.
- If x is even and y is odd, let $x = 2p$ and $y = 2q + 1$. Then, $m = (2p)^2 + 3(2q + 1)^2 = 4p^2 + 3(4q^2 + 4q + 1) = 2(2p^2 + 6q^2 + 6q + 1) + 1$, so m is odd.

Therefore, either 4 divides m or m is odd, so the contrapositive, and by extension the original statement, is true. \square

- (e) The sum of a rational number and an irrational number is irrational.

Proof. First, recall that rational numbers are those which can be expressed by $\frac{p}{q}$ for integers p and q .

Let x be a rational number and suppose for a contradiction that y is irrational such that $x + y$ is rational.

Then, $x = \frac{p}{q}$ for integers p and q . Also, $x + y = \frac{n}{m}$ for integers n and m . Substituting, $\frac{p}{q} + y = \frac{n}{m}$. Rearranging,

$$\frac{p}{q} + y = \frac{n}{m} \iff p + yq = \frac{qn}{m} \iff y = \frac{qn - mp}{qm}$$

but if y equals the ratio of two integers ($qn - mp$ and qm), by definition, y is rational.

Therefore, by contradiction, the sum of a rational number and an irrational number is irrational. \square

- (f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

Proof. Let x be a non-zero real such that $x + \frac{1}{x} < 2$. Then,

$$\begin{aligned}x + \frac{1}{x} &< 2 \\x + \frac{1}{x} - 2 &< 0 \\ \frac{x^2 + 1 - 2x}{x} &< 0 \\ \frac{(x-1)^2}{x} &< 0\end{aligned}$$

Because $(x-1)^2$ is a square, so is always non-negative, $\frac{1}{x} < 0$, which is true if and only if $x < 0$. \square

Challenges

C01. Let n be an integer. Prove that if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

Proof. Let n be an integer such that $2 \mid n$ and $3 \mid n$. Then, there exist integers $2p = n$ and $3q = n$. Equivalently, $6p = 3n$ and $6q = 2n$. Subtracting, $n = 6(p - q)$, and since $p - q$ is an integer, $6 \mid n$. \square

C02. Let $a, b, c \in \mathbb{R}$. Prove that if $a^2 + b^2 + c^2 = 1$, then $-1/2 \leq ab + bc + ca \leq 1$.

Proof. Let a, b , and c be real numbers. Recall that squares of reals are non-negative. Then, notice that we can create $2ab$ -type terms in squares of binomials:

$$\begin{aligned}0 &\leq (a-b)^2 + (b-c)^2 + (c-a)^2 \\ 0 &\leq 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \\ 0 &\leq (a^2 + b^2 + c^2) - (ab + bc + ca) \\ ab + bc + ca &\leq 1\end{aligned}$$

Likewise, we can create these terms in the square of a trinomial:

$$\begin{aligned}0 &\leq (a+b+c)^2 \\ 0 &\leq a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\ -(a^2 + b^2 + c^2) &\leq 2(ab + bc + ca) \\ -\frac{1}{2} &\leq ab + bc + ca\end{aligned}$$

Therefore, combining these inequalities, $-\frac{1}{2} \leq ab + bc + ca \leq 1$, as desired. \square

C03. Show that if p and $p^2 + 2$ are prime, then $p^3 + 2$ is also prime.

C04. Express the following statement in symbolic form and prove that it is true.

There exists a real number L such that for every positive real number ϵ , there exists a positive real number δ such that for all real numbers x , if $|x| < \delta$, then $|3x - L| < \epsilon$.

Proof. In symbolic form, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$:

$$\exists L \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x| < \delta \implies |3x - L| < \epsilon$$

We propose $L = 0$. Let $\epsilon > 0$. Select $\delta = \frac{\epsilon}{3}$. Now, suppose that $|x| < \delta$. Then, $|3x - L| = |3x| = 3|x| < 3\delta = 3\frac{\epsilon}{3} = \epsilon$, as desired. \square

C05. Prove that there are no positive integers a and b such that $b^4 + b + 1 = a^4$.

C06. Prove that the length of at least one side of a right-angled triangle with integer side lengths must be divisible by 3.