MATH 135 Fall 2020: Extra Practice 5

Warm-Up Exercises

WE01. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 9\}$, and $B = \{4, 5, 6, 7\}$.

(a) Calculate the following:

i. $A \cup B = \{2, 4, 5, 6, 7, 9\}$
ii. $A \cap B = \{4, 6\}$
iii. $\overline{A} = \mathcal{U} - A = \{1, 3, 5, 7, 8\}$
iv. $\overline{B} = \mathcal{U} - B = \{1, 2, 3, 8, 9\}$
v. $A - B = \{2, 9\}$
vi. $B - A = \{5, 7\}$

(b) Are A and B disjoint? *No, since 4 is in both sets.*

(c) Give a set C such that $C \subseteq B$. *Let $C = B$.*

(d) Give a set D such that $D \subsetneq A$. *Let $D = \{2\}$.*

WE02. Suppose S and T are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?

Proof. Let S and T be arbitrary sets such that their intersection is S. We must show that any element of S is an element of T.

Consider an element s in S. But S is equal to $S \cap T$. Elements of an intersection are elements of the original sets, so $s \in T$, as desired.

For the converse, consider another two sets, S_1 and T_1, where $S_1 \subseteq T_1$. This means that all elements of S_1 are elements of T_1, that is, all elements of S_1 are elements of both S_1 and T_1. But this is just the definition of the intersection of S_1 and T_1. Therefore, the converse is also true. □

WE03. Give an example of three sets A, B, and C such that $B \neq C$ and $B - A = C - A$.

Solution. Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$. Then, $B - A = \{2\}$ and $C - A = \{2\}$. □

Recommended Problems

RP01. Let A be a subset of the universe $\mathcal{U}$. Prove that $A \cup \overline{A} = \mathcal{U}$.

Proof. Recall that the complement of a set $\overline{S}$ with respect to a universe $\mathcal{U}$ is defined as the set $\{x \in \mathcal{U} : \neg(x \in S)\}$. Recall also that the union of two sets X and Y, again with universe $\mathcal{U}$, is defined as the set $X \cup Y = \{x \in \mathcal{U} : x \in X \vee x \in Y\}$.

Then, $A \cup \overline{A} = \{x \in \mathcal{U} : x \in A \vee \neg(x \in A)\}$. The disjunction of any logical statement with its negation is a tautology, so this property is true for all elements of $\mathcal{U}$. Therefore, the resulting set is simply $\mathcal{U}$. □

RP02. Let S and T be two sets which are subsets of the universe $\mathcal{U}$. Prove that

$$(S \cup T) - (S \cap T) = (S - T) \cup (T - S).$$

Proof. Let S and T be arbitrary subsets of $\mathcal{U}$, and x be an arbitrary element of $\mathcal{U}$ such that it is an element of the left-hand side. We prove by showing that the left and right-hand sides are subsets of another, that is, the following universally quantified biconditional holds:

$$\forall x \in \mathcal{U}, x \in (S \cup T) - (S \cap T) \iff x \in (S - T) \cup (T - S)$$

This can be done by rewriting both sides in set-builder notation and applying logical equivalencies.

$$\begin{aligned}(S \cup T) - (S \cap T) &= \{x \in \mathcal{U} : (x \in S \cup T) \wedge (x \notin S \cap T)\}\\ &= \{x \in \mathcal{U} : (x \in S \vee x \in T) \wedge \neg(x \in S \wedge x \in T)\}\\ &= \{x \in \mathcal{U} : (x \in S \vee x \in T) \wedge (\neg(x \in S) \vee \neg(x \in T))\}\end{aligned}$$

Now, distributing, we have the property:

$$(x \in S \wedge x \notin S) \vee (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S) \vee (x \in T \wedge x \notin T)$$

which can be equivalently expressed by removing falsities:

$$(x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S).$$

Now, we can apply the definitions of unions and complements in reverse:

$$\begin{aligned}(S \cup T) - (S \cap T) &= \{x \in \mathcal{U} : (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)\}\\ &= \{x \in \mathcal{U} : (x \in S \wedge x \notin T)\} \cup \{x \in \mathcal{U} : (x \in T \wedge x \notin S)\}\\ &= (S - T) \cup (T - S)\end{aligned}$$

□

RP03. Let $A = \{n \in \mathbb{Z} : 2 \mid n\}$ and $B = \{n \in \mathbb{Z} : 4 \mid n\}$. Let $n \in \mathbb{Z}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer k.

Proof. We prove the biconditional by proving both implications.

($\Rightarrow$) Let n be an arbitrary integer element of $A - B$, i.e., $n \in A$ but $n \notin B$. Then, the defining property of A holds but that of B does not. Therefore, $2 \mid n$ but $4 \nmid n$.

Since $2 \mid n$, it may be written as $n = 2q$ for some integer q.

If q is even, then $n = 2(2p)$ for some integer p. That means $n = 4p$, so $n \mid 4$, which is a contradiction. Therefore, q must be odd, and n may be written as $n = 2k$ for an odd integer $k = q$.

($\Leftarrow$) Let n be an arbitrary integer such that $n = 2k$ for some odd integer k. It immediately follows that $2 \mid n$ and $n \in A$.

Also, since k is odd, $n = 2(2d + 1) = 4\left(d + \frac{1}{2}\right)$ for another integer d. $d + \frac{1}{2}$ will never be an integer, so $4 \nmid n$, which means $n \notin B$.

However, $n \in A$ and $n \notin B$ is precisely the definition of $n \in (A - B)$.

Therefore, since both implications hold, the statement is true. □

RP04. Prove that there exist sets A, B, and C such that $A \cup B = A \cup C$ and $B \neq C$.

Proof. Let $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. Clearly, $B \neq C$.

We have $A \cup B = \{1, 2\}$ and $A \cup C = \{1, 2\}$, which are equal. □

RP05. Prove or disprove. If A, B, and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup C$.

Solution. Let A, B, and C be arbitrary sets. We disprove by showing $(A \cap B) \cup C$ is not a subset of $A \cap (B \cup C)$.

Let x be an element of C that is not an element of A. Then, it is clearly an element of $(A \cap B) \cup C$, since it is an element of C and all elements of either set in a union are elements of the union.

However, it is not an element of $A \cap (B \cup C)$. Since it is an intersection, all such elements are elements of A, which x is not.

Therefore, $(A \cap B) \cup C \not\subseteq A \cap (B \cup C)$. Set equality is defined by bidirectional subsets, so the sets cannot be equal. □

RP06. Prove there is a unique set T such that for every set S, $S \cup T = S$.

Proof. We suppose that $T = \emptyset$, that is, T is the set with no elements, and prove it.

(Existence) Since there are no elements in T, it may be written as $T = \{x : P\}$ where P is a false logical statement.

Now, the union $S \cup T$ is $\{x : x \in S \vee P\}$. but a statement disjoined with false gives itself, so we have $\{x : x \in S\}$, which is just S.

(Uniqueness) Let A and B be empty sets.

Then, $\forall x \in \mathcal{U}, x \in A \implies x \in B$ is vacuously true, since the hypothesis is always false by definition. Therefore, $A \subseteq B$.

Likewise, $\forall x \in \mathcal{U}, x \in B \implies x \in A$ is vacuously true. Therefore, $B \subseteq A$.

Since both A and B are mutual subsets, $A = B$, and the empty set is unique. □

Challenges

C01. The *symmetric difference* of two sets A and B, denoted $A \triangle B$, is defined as

$$A \triangle B = (A - B) \cup (B - A).$$

Prove that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Proof. We will prove using logical equivalences.

Consider the left-hand side. By the given definition,

$$\begin{aligned}(A \triangle B) \triangle C &= ((A - B) \cup (B - A)) \triangle C \\ &= (((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A)))\end{aligned}$$

which is a mess, so we re-express as a logical expression in set-builder notation. That is, $\{x : P(x)\}$ for some open sentence $P(x)$. For convenience, let $a \equiv x \in A$, $b \equiv x \in B$, and $c \equiv x \in C$.

$$\begin{aligned}
P(x) &\equiv (((a \wedge \neg b) \vee (b \wedge \neg a)) \wedge \neg c) \vee (c \wedge \neg((a \wedge \neg b) \vee (b \wedge \neg a))) \\
&\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge \neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a)) \\
&\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge (\neg a \vee b) \wedge (\neg b \vee a))
\end{aligned}$$

We now digress from this (also enormous) expression to simplify the last term. Recall in Problem RP02, we proved $(X \vee Y) \wedge (\neg X \vee \neg Y) \equiv (X \wedge Y) \vee (\neg X \wedge \neg Y)$. We may now apply this equivalence with $X \equiv \neg a$ and $Y \equiv b$.

$$\begin{aligned}
P(x) &\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge ((\neg a \wedge b) \vee (\neg b \wedge a))) \\
&\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge \neg a \wedge b) \vee (c \wedge \neg b \wedge a) \\
&\equiv (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)
\end{aligned}$$

Now, consider the right-hand side. By the given definition,

$$\begin{aligned}
A \bigtriangleup (B \bigtriangleup C) &= A \bigtriangleup ((B - C) \cup (C - B)) \\
&= (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)
\end{aligned}$$

which we may express as $\{x : Q(x)\}$ for some open sentence $Q(x)$.

$$\begin{aligned}
Q(x) &\equiv (a \wedge \neg((b \wedge \neg c) \vee (c \wedge \neg b))) \vee (((b \wedge \neg c) \vee (c \wedge \neg b)) \wedge \neg a) \\
&\equiv (a \wedge (\neg(b \wedge \neg c) \wedge \neg(c \wedge \neg b))) \vee ((b \wedge \neg c \wedge \neg a) \vee (c \wedge \neg b \wedge \neg a)) \\
&\equiv (a \wedge (\neg b \vee c) \wedge (\neg c \vee b)) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)
\end{aligned}$$

Applying the identity we just discovered, namely, $X \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y) \equiv (X \wedge Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z)$, for $X \equiv a$, $Y \equiv b$, and $Z \equiv c$.

$$Q(x) \equiv (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$$

but this is exactly $P(x)$. Therefore, the right-hand side may be expressed $\{x : P(x)\}$, which is precisely the left-hand side. □