**MATH 135 Fall 2019: Midterm Examination**

**Q01**. Let $A$ and $B$ be statement variables.

(a) Complete the truth table below.

| $A$ | $B$ | $\neg B$ | $A \wedge (\neg B)$ | $B \iff A$ | $(A \wedge (\neg B)) \vee (B \iff A)$ | $B \implies A$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |

(b) Determine whether $(A \wedge (\neg B)) \vee (B \iff A)$ is logically equivalent to $B \implies A$. Circle the correct answer. No further justification is needed.

[Equivalent] Not Equivalent

**Q02**. Let $x, y \in \mathbb{R}$. Consider the implication $S$:

If $xy > 6$, then $x > 2$ and $y > 3$.

(a) State the hypothesis of $S$.

$$xy > 6$$

(b) State the conclusion of $S$.

$$x > 2 \text{ and } y > 3$$

(c) State the converse of $S$.

If $x > 2$ and $y > 3$, then $xy > 6$.

(d) State the contrapositive of $S$.

If $x \leq 2$ or $y \leq 3$, then $xy \leq 6$.

(e) State the negation of $S$ in a form that does not contain an implication.

$xy \leq 6$, $x > 2$, and $y > 3$.

(f) Indicate clearly whether the given implication $S$ is true or false for all $x, y \in \mathbb{R}$. Then prove or disprove the statement.

Circle the correct answer: True [False]

*Proof.* Suppose for a counterexample that $x = 1$ and $y = 7$. Clearly, $xy = 7 > 6$, so the hypothesis holds. However, $1 \not> 2$. Since the hypothesis is true and the conclusion false, the implication is false. □

**Q03**. Given a variable $x$, let $P(x)$ denote the open sentence $x \geq 0$, and let $Q(x)$ denote the open sentence $x < 0$. Determine if the following statements are true or false. Circle the correct answers. No justification is needed.

(a) $(\forall x \in \mathbb{R}, P(x)) \vee (\forall x \in \mathbb{R}, Q(x))$
True **[False]**

(b) $\forall x \in \mathbb{R}, (P(x) \vee Q(x))$
**[True]** False

(c) $(\forall x \in \mathbb{N}, P(x)) \vee (\forall x \in \mathbb{N}, Q(x))$
**[True]** False

(d) $\forall x \in \mathbb{N}, (P(x) \vee Q(x))$
**[True]** False

**Q04**. For each of the following statements, indicate clearly whether the statement is true or false and then prove or disprove the statement.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + 2y = 0$.
Circle the correct answer: **[True]** False

*Proof.* Let $x$ be an arbitrary real. Select $y = -\frac{x}{2}$, which is a real number. Then, $x + 2y = x + 2(-\frac{x}{2}) = x - x = 0$, as desired. □

(b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + 2y = 0$.
Circle the correct answer: True **[False]**

*Proof.* Consider the negation: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + 2y \neq 0$.

Let $y$ be an arbitrary real number. Select $x = -2y + 1$, which is a real number. Then, $x + 2y = (-2y + 1) + 2y = 1 \neq 0$, as desired.

Since the negation is true, the original statement is false. □

**Q05**. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers defined by

$$a_1 = 1, \quad a_2 = 5, \quad a_m = a_{m-1} + 2a_{m-2}, \text{ if } m \geq 3.$$

Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

*Proof.* We will prove by strong induction of $P(n)$, that $a_n = 2^n + (-1)^n$, on $n$.

To verify the base cases $P(1)$ and $P(2)$, notice that $a_1$ is defined to be 1 and $2^1 + (-1)^1 = 2 - 1 = 1$. Likewise, $a_2$ is defined to be 5 and $2^2 + (-1)^2 = 4 + 1 = 5$.

Now, suppose that for some $k \geq 3$, $P(n)$ holds for all $n < k$. Specifically, $P(k-1)$ and $P(k-2)$ hold. Then, we take the definition of $a_k$:

$$\begin{aligned}
a_k &= a_{k-1} + 2a_{m-2} \\
&= (2^{k-1} + (-1)^{k-1}) + 2(2^{k-2} + (-1)^{k-2}) && \text{by inductive hypothesis} \\
&= 2^{k-1} + (-1)^{k-1} + 2^{k-1} + 2(-1)^{k-2} \\
&= 2^k + (-1)(-1)^{k-2} + 2(-1)^{k-2} \\
&= 2^k + (-1+2)(-1)^{k-2}(-1)(-1) \\
&= 2^k + (-1)^k
\end{aligned}$$

which is exactly $P(k)$.

Therefore, by the principle of strong induction, $P(n)$ holds for all $n \geq 1$. $\square$

**Q06**.

(a) Let $n \in \mathbb{N}$. Evaluate the sum $\sum_{i=0}^{n} \binom{n}{n-i} 2^i$.

*Solution.* Recall that $1^k = 1$ for any real $k$. Then,

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{n-i} 2^i &= \sum_{i=0}^{n} \binom{n}{n-i} 2^i 1^{n-i} \\ &= (2+1)^n && \text{by binomial theorem} \\ &= 3^n \end{aligned}$$

$\square$

(b) Determine the coefficient of $x^{22}$ in the expansion of $\left(x^2 + \frac{2}{x}\right)^{14}$.

*Solution.* Apply the binomial theorem:

$$\begin{aligned} \left(x^2 + \frac{2}{x}\right)^{14} &= \sum_{i=0}^{14} \binom{14}{14-i} (x^2)^i (2x^{-1})^{14-i} \\ &= \sum_{i=0}^{14} \binom{14}{14-i} 2^{14-i} x^{2i} x^{-(14-i)} \\ &= \sum_{i=0}^{14} \binom{14}{14-i} 2^{14-i} x^{3i-14} \end{aligned}$$

The exponent on $x$, $3i - 14$, will be 22 when $i = 12$. On this term, the coefficient will be $\binom{14}{14-12} 2^{14-12} = \binom{14}{2} 2^2 = 364$. $\square$

**Q07**. Let $a, b \in \mathbb{Z}$ with $a \geq 2$. Prove that if $a \neq 13$, then $a \nmid (3b+1)$ or $3a \nmid (7b-2)$.

*Proof.* Let $a \geq 2$ and $b$ be integers. We will prove the contrapositive:

If $a \mid (3b+1)$ and $3a \mid (7b-2)$, then $a = 13$.

Suppose that $a$ divides $3b+1$ and $3a$ divides $7b-2$.

Then, we may write $3b+1 = ak$ for some $k \in \mathbb{Z}$. But this means $9b+3 = (3a)k$, so $3a \mid 9b+3$. By DIC, it follows that $3a \mid [7(9b+3) - 9(7b-2)]$. That is, $3a \mid 39$.

This means we may write $39 = 3an$ for an integer $n$. It follows that $13 = an$, so $a \mid 13$. However, 13 is prime, therefore, the only value for $a \geq 2$ is 13. $\square$

**Q08**.

(a) Prove that for all $n \in \mathbb{N}$, $n^2 + n + 1$ is odd.

*Proof.* Let $n$ be a natural number. Recall that all natural numbers are either even or odd.

If $n$ is even, we may write $n = 2k$ for some integer $k$. Then, $n^2+n+1 = 4k^2+2k+1 = 2(2k^2+k)+1$. Since $2k^2+k$ is an integer, this is an odd number.

Likewise, if $n$ is odd, we may write $n = 2k+1$ for some integer $k$. Then, $n^2+n+1 = 4k^2+4k+1+2k+1+1 = 2(2k^2+3k+1)+1$. Since $2k^2+3k+1$ is an integer, this is also an odd number.

Therefore, the sum $n^2+n+1$ is odd for all natural $n$. □

(b) Let $d, n \in \mathbb{N}$. Prove that if $d \mid (n^2+n+1)$ and $d \mid (n^2+n+3)$, then $d = 1$.

*Proof.* Let $d$ and $n$ be natural numbers such that $d$ divides both $n^2+n+1$ and $n^2+n+3$.

Consider the case that $n = 0$. We have that $d \mid 1$ and $d \mid 3$.

Also, by DIC, $d \mid \big((n^2+n+3)-(n^2+n+1)\big)$, that is, $d \mid 2$.

However, 2 and 3 are both prime, so their only divisors are 1 and themselves. Since $d$ cannot simultaneously be 2 and 3, it must be 1. □

**Q09**. Let $a$ and $b$ be non-negative integers. Prove that $3^a = 8^b$ if and only if $a = b = 0$.

*Proof.* We will prove by considering both implications.

($\Rightarrow$) Let $a$ and $b$ be non-negative integers such that $3^a = 8^b$. Recall that integers have unique prime factorizations, and that 2 and 3 are primes.

Notice that $8^b = 2^{3b}$. For any positive $a$ and $b$, because $3^a$ consists only of 3's, and $2^{3b}$ consists only of 2's, they cannot be equal. Therefore, the only possible values are $a = b = 0$. In this case, both values equal 1, so the equality holds.

($\Leftarrow$) Let $a = b = 0$. Then, $3^0 = 1$ and $8^0 = 1$, so the equality holds.

Therefore, both implications hold, so the if and only if statement is true. □

**Q10**. Prove that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}.$$

*Proof.* We will prove by inducting the statement $P(n)$, $\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$, on $n$.

To verify the base case $P(1)$, notice that

$$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{1^2} = 1$$

and that

$$2 - \frac{1}{1} = 2 - 1 = 1$$

We have $1 \le 1$, which is true.

Now, suppose that $P(k)$ holds for an arbitrary $k \geq 1$. That is,

$$\begin{aligned}
\sum_{i=1}^{k} \frac{1}{i^2} &\leq 2 - \frac{1}{k} \\
\frac{1}{(k+1)^2} + \sum_{i=1}^{k} \frac{1}{i^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
\sum_{i=1}^{k+1} \frac{1}{i^2} &\leq 2 - \frac{k^2+k+1}{k(k+1)^2} \\
&< 2 - \frac{k^2+k}{k(k+1)^2} \\
&= 2 - \frac{k(k+1)}{k(k+1)^2} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

which is $P(k+1)$.

Therefore, by the principle of mathematical induction, $P(n)$ holds for all natural $n$. $\square$