MATH 135 Winter 2020: Midterm Examination

Q01. Let P and Q be logical statements.

(a) Complete the following truth table:

P	Q	$Q \implies P$	$P \wedge (Q \implies P)$	$(Q \Rightarrow P) \iff [P \wedge (Q \Rightarrow P)]$	$P \vee Q$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	F

(b) Are the expressions $(Q \implies P) \iff [P \wedge (Q \implies P)]$ and $P \vee Q$ logically equivalent? Circle one of the options below. No justification required.

$\boxed{\text{Equivalent}}$ Not equivalent

Q02. Let x and y be real numbers. Consider the following implication S:

If x is irrational, then y is rational or xy is irrational

(a) State the hypothesis of S.

x is irrational

(b) State the conclusion of S.

y is rational or xy is irrational

(c) State the converse of S.

If y is rational or xy is irrational, then x is irrational

(d) State the contrapositive of S.

If y is irrational and xy is rational, then x is rational

(e) State the negation of S in a form that does not contain an implication.

x and y are irrational but xy is rational

(f) Indicate whether the statement $\forall x, y \in R$, S is true or false by circling one of the options below. Then either prove or disprove the statement.

Circle the correct answer: True $\boxed{\text{False}}$

Proof. We prove the negation, $\exists x, y \in \mathbb{R}$ with x and y irrational but xy rational. Consider $x = y = \sqrt{2}$. Then, x and y are irrational. However, $xy = 2$ which is rational.

Therefore, since the negation is true, the original statement is false. $\square$

Q03. Let $A = \{2k : k \in \mathbb{Z}\}$ and $B = \{2m + 1 : m \in \mathbb{Z}\}$. In (a) and (b), indicate whether each statement is true or false by circling one of the options. Then either prove or disprove the statement.

(a) $\forall a \in A, \exists b \in B, a + b = 7$
$\boxed{\text{True}}$ False

Proof. Let a be an element of A, that is, an even integer. Then, there exists a k such that $a = 2k$.

Select $b = -2k + 7$. This is equal to $2(-k + 3) + 1$, where $-k + 3$ is an integer. It follows that $b \in B$.

Now, $a + b = 2k - 2k + 7 = 7$, as desired. □

(b) $\exists b \in B, \forall a \in A, a + b = 7$
True $\boxed{\text{False}}$

Proof. Consider the negation:

$$\forall b \in B, \exists a \in A, a + b \neq 7.$$

Let b be an element of B, that is, an odd integer. Then, there exists an m such that $b = 2m + 1$.

If $m = 3$, so $b = 7$, then select $a = 2$ (i.e. $2k$ where $k = 1$) and notice $a + b = 9 \neq 7$. If $m \neq 3$, so $b \neq 7$ then select $a = 0$ (i.e. $2k$ where $k = 0$). In this case, $a + b = b \neq 7$.

Therefore, since the negation is true, the original statement is false. □

Q04.

(a) Prove that for all integers a, if $a \nmid 1$, then $a \nmid 9$ or $a \nmid 17$.

Proof. Consider the contrapositive: if $a \mid 9$ and $a \mid 17$, then $a \mid 1$.

Let a be an integer that divides both 9 and 17. Then, a also divides the integer combination $9(2) - 17(1) = 1$, as desired.

Because the contrapositive is true, the original statement is also true. □

(b) Prove that for all integers a, b, and c, if $a \mid (b + c)$, then $a^2 \nmid (2b + 3c)$ or $a \mid c$.

Proof. Let a, b, and c be integers.

Consider the negation, that $a \mid (b + c)$, $a^2 \mid (2b + 3c)$, and $a \nmid c$. Suppose for a contradiction that the negation is true.

Clearly, $a \mid a^2$, so by TD, $a \mid (2b + 3c)$. But we have $a \mid (b + c)$, so this means that by DIC, $a \mid ((2b + 3c) - 2(b + c))$. This is just $a \mid c$, which is a contradiction.

Therefore, the negation is false, so the original statement must be true. □

Q05. Let a, b, and c be odd integers. Prove that there does not exist a right triangle with side lengths a, b, and c.

Proof. Suppose for a contradiction that such a right triangle exists.

Then, there exist a, b, and c such that $a^2 + b^2 = c^2$. Since they are odd, we may write a, b, and c as $2r+1$, $2s+1$, and $2t+1$, respectively.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (2r+1)^2 + (2s+1)^2 &= (2t+1)^2 \\ 4r^2 + 4r + 1 + 4s^2 + 4s + 1 &= 4t^2 + 4t + 1 \\ 2(2r^2 + 2r + 2s^2 + 2s + 1) &= 2(2t^2 + 2t) + 1 \end{aligned}$$

Since $2r^2 + 2r + 2s^2 + 2s + 1$ and $2t^2 + 2t$ are both integers, the left-hand side represents an even integer and the right-hand side represents an odd integer. There are no integers that are both even and odd, so this is a contradiction.

Therefore, there is no right triangle with three odd side lengths. □

Q06. Let a be an integer. Prove that if $4 \mid a(a+x)$ for all integers x, then $4 \mid a$.

Proof. Let a be an integer such that $4 \mid a(a+x)$ for all integers x.

The statement must be true for all integers x, and $-a+1$ is an integer, so we may let $x = -a+1$. Then, $4 \mid a(a-a+1)$, which is just $4 \mid a$. □

Q07.

(a) Determine the coefficient of x^4 in the expansion of $\left(2x^{14} + \dfrac{1}{x^3}\right)^{10}$

Solution. Apply the binomial theorem:

$$\begin{aligned} \left(2x^{14} + \frac{1}{x^3}\right)^{10} &= \sum_{k=0}^{10} \binom{10}{k} (2x^{14})^k \left(\frac{1}{x^3}\right)^{10-k} \\ &= \sum_{k=0}^{10} \binom{10}{k} 2^k x^{14k} x^{-3(10-k)} \\ &= \sum_{k=0}^{10} \binom{10}{k} 2^k x^{17k-30} \end{aligned}$$

The exponent on x will be 4 when $17k - 30 = 4$, that is, $k = 2$. Here, the coefficient is $\binom{10}{2} 2^2 = 45 \cdot 4 = 180$. □

(b) Evaluate the sum $\displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{3^{2i}\, 5^{n-i}}{2^{3i}}$

Solution. Rearrange terms to match the binomial theorem and apply it:

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} \frac{3^{2i}\, 5^{n-i}}{2^{3i}} &= \sum_{i=0}^{n} \binom{n}{i} \frac{(3^2)^i}{(2^3)^i} 5^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{9}{8}\right)^i 5^{n-i} \\ &= \left(\frac{9}{8} + 5\right)^n \\ &= \left(\frac{49}{8}\right)^n \end{aligned}$$

or, expressed similarly to the original, $\frac{7^{2n}}{2^{3n}}$. □

Q08.

(a) Let A, B, and C be sets. Prove that if $A - B \subseteq C$, then $A - (B \cup C) = \emptyset$.

Proof. Let A, B, and C be sets such that $A - B \subseteq C$. Let a be an element of A.

Suppose for a contradiction that a is in neither B nor C. Then, because $a \notin B$, it is in $A - B$. However, $A - B \subseteq C$, so $a \in C$. This is a contradiction.

Therefore, a is in either B or C, that is, it is in $B \cup C$. Thus, since all elements of A are in $B \cup C$, the set difference is the empty set. □

(b) Consider the sets

$$A = \{n \in \mathbb{N} : n \geq 2\}, \quad B = \{a \in A : 3 \mid (2a + 1)\}, \quad C = \{(2k + 5)^2 : k \in \mathbb{Z}\}.$$

Prove that $B \cap C \neq \emptyset$.

Proof. It suffices to show the existence of an element of $B \cup C$. We propose $x = 49$ and prove it.

Clearly, $49 \geq 2$, so $x \in A$. Also, $2x + 1 = 99$, a multiple of 3. Therefore, $x \in B$.

Let $k = 1$, which is an integer. Then, $(2k + 5)^2 = 7^2 = 49$. Therefore, $x \in C$.

Since x is an element in both B and C, it is in their intersection. As we have proven that the size of $B \cup C$ is at least 1, it is not the empty set. □

Q09. The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all integers $n \geq 2$. Prove that for every non-negative integer n,

$$\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}.$$

Proof. Let $P(n)$ be the statement that $\sum_{i=0}^{n} f_i^2 = f_n f_{n-1}$. We will prove by induction.

For the base case $P(0)$, notice that

$$\sum_{i=0}^{0} f_i^2 = f_0^2 = 0 = 0 \cdot 1 = f_0 f_1$$

Now, suppose that for an arbitrary non-negative integer k, $P(k-1)$ holds. Then,

$$\begin{aligned}
\sum_{i=0}^{k} f_i^2 &= f_k^2 + \sum_{i=0}^{k-1} f_i^2 \\
&= f_k^2 + f_{k-1} f_k && \text{by inductive hypothesis} \\
&= f_k(f_k + f_{k-1}) \\
&= f_k f_{k+1}
\end{aligned}$$

which is exactly $P(k)$.

Therefore, by induction, $P(n)$ holds for all non-negative integers. □

Q10. Prove that for all $n \in \mathbb{N}$, there exist non-negative integers a and b such that $3 \nmid b$ and $n = 3^a b$.

Proof. We will strongly induct the statement $P(n)$, that a and b exist such that $3 \nmid b$ and $n = 3^a b$, on n.

For the base cases $P(1)$ and $P(2)$, let $a = 0$ and $b = n$. Then, $3 \nmid n$ and $3^a b = b = n$. For the base case $P(3)$, let $a = 1$ and $b = 1$. Then, $3 \nmid 1$ and $3^a b = 3$.

Let $m \geq 4$ be an arbitrary integer. Suppose that for all integers $n < m$, $P(n)$ holds.

If $3 \mid m$, then there exists an integer k such that $m = 3k$ we use the fact that $P(k)$ holds. This ensures that there exist a_0 and b_0 such that $3^{a_0} b_0 = k$. Rearranging,

$$\begin{aligned} 3^{a_0} b_0 &= k \\ 3^{a_0} b_0 &= \frac{m}{3} \\ m &= 3^{a_0+1} b_0 \end{aligned}$$

which, with $a = a_0 + 1$ and $b = b_0$, is exactly what we need to show to prove $P(m)$. $\square$