MATH 135 Fall 2020:
Extra Practice

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Try the problems first before looking at solutions. You won't learn by reading someone else's work.

Chapter 1

Introduction to the Language of Mathematics

1.1 Warm-Up Exercises

Warm-Up Exercise 1.1. Determine if the following quantified statements are true or false. No justification is needed.

(a) $\forall x \in \mathbb{R}, \sin^2 x + \cos^2 x = 1$

(b) $\exists y \in \mathbb{Z}, 6y - 3 = 28$

(c) $\forall p \in \mathbb{Q}, \exists q \in \mathbb{Z}, |p - q| \leq 1$

Solution. (a) True, by the Pythagorean identity.

(b) False, since $6y - 3 = 28 \implies 6y = 31 \implies y = \frac{31}{6}$, which is undefined in $\mathbb{Z}$.

(c) True, select $q = \lfloor p \rfloor$. □

1.2 Recommended Problems

Recommended Problem 1.1. Which of the following are statements? If it is a statement, determine if it is true or false. No justification is needed.

(a) $3 \leq \pi$

(b) $2x - 3 \geq -1$

(c) $x^2 - y^3 = 1$

(d) N is a perfect square.

(e) $x^2 + 5x - 2$

(f) $x \leq x + 1$

(g) There is a largest real number.

(h) There is a smallest positive number.

(i) Every real number is either positive or negative.

(j) Some triangles are right triangles.

Solution. (a) Statement, true.

(b) Not a statement, depends on x.

(c) Not a statement, depends on x and y.

(d) Not a statement, depends on N.

(e) Not a statement or an open sentence.

(f) Not a statement, depends on x.

(g) Statement, false.

(h) Statement, true.

(i) Statement, false.

(j) Statement, true. □

Recommended Problem 1.2. For each of the following statements, identify the four parts of the quantified statement (quantifier, variables, domain, and open sentence). Next, express the statement in symbolic form using as few words as possible and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq$ are okay). Finally, determine if the original statement is true or false. No justification is needed.

(a) The equation $x^2 + 2x - 3 = 0$ has a real solution.

(b) No matter which real value x we choose, $-1 + \cos x$ will always be positive.

(c) Every natural number can be expressed as the product of two integers.

(d) There is a perfect square which is also a perfect cube.

Solution. (a)

$$\exists x \in \mathbb{R}, x^2 + 2x - 3 = 0$$

Quantifier: existential; variable: x; domain: $\mathbb{R}$; open sentence: $x^2 + 2x - 3 = 0$. Negation:

$$\forall x \in \mathbb{R}, x^2 + 2x - 3 \neq 0$$

The statement is *true*.

(b)
$$\forall x \in \mathbb{R}, -1 + \cos x > 0$$
Quantifier: universal; variable: x; domain: $\mathbb{R}$; open sentence: $-1 + \cos x > 0$. Negation:
$$\exists x \in \mathbb{R}, -1 + \cos x \leq 0$$
The statement is *false*.

(c)
$$\forall n \in \mathbb{N}, \exists a, b \in \mathbb{Z}, n = ab$$
Quantifier: universal/existential; variables: n, a, b; domain: $\mathbb{N}$, $\mathbb{Z}$; open sentence: $n = ab$. Negation:
$$\exists n \in \mathbb{N}, \forall a, b \in \mathbb{Z}, n \neq ab$$
The statement is *true*.

(d)
$$\exists n \in \mathbb{Z}, \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, n = x^2 = y^3$$
Quantifier: existential; variables: n, x, y; domain: $\mathbb{Z}$; open sentence: $n = x^2 = y^3$. Negation:
$$\forall n \in \mathbb{Z}, \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, n \neq x^2 \wedge n \neq y^2$$
The statement is *true*. □

Recommended Problem 1.3. Negate the following statements without using words or the $\neg$ symbol. For each statement determine whether it or its negation is true.

(a) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, 3a = b$

(b) $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, 3a = b$

(c) $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \exists c \in \mathbb{R}, \frac{a}{c} = b$

Solution. (a) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, 3a = b$: false
$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, 3a \neq b$: true

(b) $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, 3a = b$: true
$\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, 3a \neq b$: false

(c) $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \exists c \in \mathbb{R}, \frac{a}{c} = b$: true
$\exists a \in \mathbb{R}, \exists b \in \mathbb{R}, \forall c \in \mathbb{R}, \frac{a}{c} \neq b$: false □

Recommended Problem 1.4. Express the following statement symbolically without using any words: *Every integer is a perfect square.*

Solution. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, n = m^2$ □

Chapter 2

Logical Analysis of Mathematical Statements

2.1 Warm-Up Exercises

Warm-Up Exercise 2.1. Let A, B and C be statement variables. Determine the truth table of $(A \wedge B) \implies \neg C$.

Solution.

A	B	C	$A \wedge B$	$\neg C$	$(A \wedge B) \implies \neg C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	F	F	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	F	T	T

□

Warm-Up Exercise 2.2. State the contrapositive and the converse of the following implication: If Jane is a doctor, then she went to medical school.

Solution. Converse: If Jane went to medical school, then she is a doctor. *Contrapositive*: If Jane did not go to medical school, then she is not a doctor. □

2.2 Recommended Problems

Recommended Problem 2.1. For each of the following statements, identify the four parts of the quantified statement (quantifier, variables, domain, and open sentence). Next, express the statement in symbolic form using as few words as possible and then write down the negation of the statement (when possible, without using any negative words such as "not" or the $\neg$ symbol, but negative math symbols like $\neq$ are okay). Finally determine if the original statement is true or false. No justification is needed.

(a) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

(b) For every even integer a and odd integer b, a rational number c can always be found such that $a < c < b$ or $b < c < a$.

Solution. (a)

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x \neq y \implies x^2 + y^2 > 0$$

Quantifier: universal; variable: x; domain: $\mathbb{R}$; open sentence: $x \neq y \Rightarrow x^2+y^2 > 0$. Negation: $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x \neq y \wedge x^2 + y^2 \leq 0$. The statement is *true*.

(b)

$$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, \exists c \in \mathbb{Q}, \left(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}\right) \implies (a < c < b \vee b < c < a)$$

Quantifier: universal/existential; variables: a, b, c; domain: $\mathbb{Z}$, $\mathbb{Q}$; open sentence: $(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}) \Rightarrow (a < c < b \vee b < c < a)$. Negation:

$$\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, \forall c \in \mathbb{Q}, \left(\frac{a}{2} \in \mathbb{Z} \wedge \frac{b-1}{2} \in \mathbb{Z}\right) \wedge ((c \leq a \vee c \geq b) \wedge (c \leq b \vee c \geq a))$$

The statement is *true*. □

Recommended Problem 2.2. Let A and B be statement variables. Prove that $(\neg A) \vee B$ is logically equivalent to $\neg(A \wedge \neg B)$.

Proof. Apply De Morgan's law: $(\neg A) \vee B \equiv \neg(A \wedge \neg B)$. □

Recommended Problem 2.3. Let A and B be statement variables. Determine whether $A \implies B$ is logically equivalent to $(\neg A) \vee B$.

Proof. $A \implies B$ is defined as $\neg(A \wedge \neg B)$. This is easily verifiable by noticing that an implication is only false when the hypothesis is true but the conclusion is false. Expand using De Morgan's law: $\neg(A \wedge \neg B) \equiv (\neg A \vee B)$. □

Recommended Problem 2.4. Assume that it has been established that the following implication is true:

If I don't see my advisor today, then I will see her tomorrow.

For each of the sentences below, determine if it is true or false. No justification is needed. If you can't determine the truth value of the sentence, explain why.

(a) I don't meet my advisor both today and tomorrow. (This is arguably an ambiguous English sentence. Answer the problem using both interpretations.)

(b) I meet my advisor both today and tomorrow.

(c) I meet my advisor either today or tomorrow (but not on both days).

Solution. (a) For the case of not today and not tomorrow, the statement is contradictory. For the case of today or tomorrow, exclusive, see (c).

(b) Not contradictory, but the truth value is indeterminate because we do not know about meeting "today".

(c) Not contradictory, but the truth value is indeterminate because we do not know about meeting "today". □

Recommended Problem 2.5. Let A, B and C be statement variables. Prove the following logical equivalence using a chain of logical equivalences as in Chapter 2.3 of the notes.

$$(A \wedge C) \vee (B \wedge C) \equiv \neg((A \vee B) \implies \neg C)$$

Proof. Begin by considering the implication on the right-hand side. Recall the definition of an implication $X \implies Y \equiv \neg X \vee Y$. Apply this and simplify:

$$\begin{aligned} \neg((A \vee B) \implies \neg C) &\equiv \neg(\neg(A \vee B) \vee \neg C) \\ &\equiv \neg(\neg(A \vee B)) \wedge \neg(\neg C) && \text{De Morgan's law} \\ &\equiv (A \vee B) \wedge C && \text{Double negation} \\ &\equiv (A \wedge C) \vee (B \wedge C) && \text{Distributive conjunction} \end{aligned}$$

Hence, the left side is logically equivalent to the right side, so the equivalency holds. □

Recommended Problem 2.6. Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

Proof. For each friend, let A, B, G, and D be if they go to the movies, respectively. We can write our statements as implications: $A \implies B$, $B \implies G$, and $G \implies D$. By the transitivity of the implication, $A \implies G$, $A \implies D$, and $B \implies D$. Recall that only two of A, B, G, and D are allowed to be simultaneously true. If A is true, then all of B, G, and D are true, which is a contradiction. Therefore, A is false. If B is true, then both G and D are true, which is a contradiction. Therefore, B is false. This leaves G (which implies D) and D to be true, which satisfies our exclusivity condition. Therefore, Gina and Dana atttended the movies. □

Recommended Problem 2.7. Consider the following statement.

$$\text{For all } x \in \mathbb{R}, \text{ if } x^6 + 3x^4 - 3x < 0, \text{ then } 0 < x < 1$$

(a) Rewrite the given statement in symbolic form.

(b) State the hypothesis of the implication within the given statement.

(c) State the conclusion of the implication within the given statement.

(d) State the converse of the implication within the given statement.

(e) State the contrapositive of the implication within the given statement.

(f) State the negation of the given statement without using the word "not" or the $\neg$ symbol (but symbols such as $\neq$, $\nmid$, etc. are fine).

Solution. (a) $\forall x \in \mathbb{R}, x^6 + 3x^4 - 3x < 0 \implies 0 < x < 1$

(b) $x^6 + 3x^4 - 3x < 0$

(c) $0 < x < 1$

(d) $0 < x < 1 \implies x^6 + 3x^4 - 3x < 0$

(e) $x \leq 0 \vee x \geq 1 \implies x^6 + 3x^4 - 3x \geq 0$

(f) $\exists x \in \mathbb{R}, x^6 + 3x^4 - 3x < 0 \wedge (x \leq 0 \vee x \geq 1)$ □

Chapter 3

Proving Mathematical Statements

3.1 Warm-Up Exercises

Warm-Up Exercise 3.1. Prove the following two quantified statements.

(a) $\forall n \in \mathbb{N}, n+1 \geq 2$

(b) $\exists n \in \mathbb{Z}, \frac{5n-6}{3} \in \mathbb{Z}$

Proof. (a) Let $n \in \mathbb{N}$. Recall that 1 is the smallest natural. $n \geq 1 \iff n+1 \geq 2$.

(b) Select $n = 3$. Then, $\frac{5n-6}{3} = \frac{15-6}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$. □

Warm-Up Exercise 3.2. Prove that for all $k \in \mathbb{Z}$, if k is odd, then $4k+7$ is odd.

Proof. Let k be an odd integer. Then, it can be written as $2n+1$ for some integer n.

Substituting, $4k+7 = 4(2n+1)+7 = 8n+11 = 2(4n+5)+1$. By definition, since $4k+7$ can be written as $2m+1$ where $m = 4n+5$ is an integer, it is odd. □

Warm-Up Exercise 3.3. Consider the following proposition

$$\textit{For all } a, b \in \mathbb{Z}, \textit{ if } a^3 \mid b^3, \textit{ then } a \mid b.$$

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

(a) Consider $a = 2$, $b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.

(b) Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, then $a^3 \mid b^3$.

(c) Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.

Solution. (a) This proof is erroneous as it only considers one specific case of a and b and not the general case of integer a and b.

(b) This proof supposes the conclusion instead of the hypothesis.

(c) The proof does not guarantee that $\frac{ka^2}{b^2}$ is an integer. □

Warm-Up Exercise 3.4. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$, then $x \neq 5$.

Proof. Suppose for the contrapositive that $x = 5$. Then, $x^3 - 5x^2 + 3x = (5)^3 - 5(5)^2 + 3(5) = 15$, as required. Since the contrapositive is true, the original implication must be true. □

Warm-Up Exercise 3.5. Prove that there do not exist integers x and y such that $2x+4y = 3$.

Proof. For the sake of contradiction, suppose the negation is true.

Consider the negation of the statement: there exist integers x and y such that $2x + 4y = 3$. Let x and y be such integers. Then, $x+2y$ is an integer. Therefore, $2x+4y = 2(x+2y)$ is even. However, 3 is odd. An integer cannot be both even and odd, therefore, the negation is false, and the original statement is true. □

Warm-Up Exercise 3.6. Prove that an integer is even if and only if its square is an even integer.

Proof. ($\Rightarrow$) Let n be an even integer. Then, $n = 2k$ for some integer k. $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

($\Leftarrow$) Let n be an even square integer. Then, $n = 2k$ for some integer k and $n = x \cdot x$ for some integer x. Since $2k = x \cdot x$, and 2 is prime, 2 must divide x. Therefore, $x = 2y$ for some integer y, which is the definition of being even.

Since the implication is true in both directions, the biconditional is true. □

3.2 Recommended Problems

Recommended Problem 3.1. Prove that $x^2 + 9 \geq 6x$ for all real numbers x.

Proof. Let x be a real number. $x^2 + 9 \geq 6x \iff x^2 - 6x + 9 \geq 0 \iff (x-3)^2 \geq 0$. Since the square of a real is always non-negative, the statements are true. □

Recommended Problem 3.2. Prove that for all $r \in \mathbb{R}$ where $r \neq -1$ and $r \neq -2$,

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r(2^r)}{(r+1)(r+2)}$$

Proof. Let r be a real number that is neither -1 nor -2. Then,

$$\begin{aligned}
LHS &= \frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} \\
&= \frac{2^{r+1}(r+1) - 2^r(r+2)}{(r+1)(r+2)} \\
&= \frac{r2^{r+1} + 2^{r+1} - r2^r - 2 \cdot 2^r}{(r+1)(r+2)} \\
&= \frac{r2^{r+1} + \cancel{2^{r+1}} - r2^r - \cancel{2^{r+1}}}{(r+1)(r+2)} \\
&= \frac{r(2^{r+1} - 2^r)}{(r+1)(r+2)} \\
&= \frac{r(2^r \cdot 2 - 2^r)}{(r+1)(r+2)} \\
&= \frac{r(2^r + \cancel{2^r} - \cancel{2^r})}{(r+1)(r+2)} \\
&= \frac{r(2^r)}{(r+1)(r+2)} \\
&= RHS
\end{aligned}$$

Since the left side equals the right side, the equality is true. □

Recommended Problem 3.3. Prove that there exists a real number x such that $x^2 - 6x + 11 \leq 2$.

Proof. Let $x = 3$. $x^2 - 6x + 11 = (3)^2 - 6(3) + 11 = 9 - 18 + 11 = 2 \leq 2$, as required. Since 3 is a real number, the statement is true. □

Recommended Problem 3.4. Prove or disprove each of the following statements.

(a) $\forall n \in \mathbb{Z}$, $\frac{5n-6}{3}$ is an integer.

(b) $\forall a \in \mathbb{Z}$, $a^3 + a + 2$ is even.

(c) For every prime number p, $p + 7$ is composite.

(d) For all $x \in \mathbb{R}$, $|x - 3| + |x - 7| \geq 10$.

(e) There exists a natural number $m < 123456$ such that $123456m$ is a perfect square.

(f) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

(a)

Proof. Let $n = 1$ as a counter-example. Then, $\frac{5n-6}{3} = \frac{5-6}{3} = -\frac{1}{3}$, which is not an integer. Therefore, the statement is false. □

(b)

Proof. Let a be an integer. Then, a is either even or odd. Suppose that a is even and can be written as $a = 2k$ for an integer k. Then, $a^3 + a + 2 = (2k)^3 + 2k + 2 = 8k^3 + 2k + 2 = 2(4k^3 + k + 1)$, an even number.

Suppose a is odd and can be written as $a = 2k + 1$ for an integer k. Then, $a^3 + a + 2 = (2k + 1)^3 + (2k + 1) + 2 = 8k^3 + 12k^2 + 8k + 4 = 2(4k^3 + 6k^2 + 4k + 2)$, an even number.

Therefore, the statement is true. □

(c)

Proof. Let p be a prime number.

If p is even, then $p = 2$, and $p + 7 = 9$ which is composite.

If p is odd, $p = 2k + 1$ for some integer $k \geq 0$ (as there are no negative primes). Then, $p + 7 = 2k + 8 = 2(k + 4)$, which is even. The only even prime is 2, but $2k + 8 \geq 8$, so $p + 7$ is composite.

Therefore, since all primes are either even or odd, $p + 7$ is composite for all primes. □

(d)

Proof. Let $x = 3$ as a counter-example. Then, $|x-3|+|x-7| = |(3)-3|+|(3)-7| = 0+4 = 4 \not\geq 10$. Therefore, the statement is false. □

(e)

Proof. Let $m = 1929$, which is a natural number less than 123456. Then, $123456m = 238146624 = 15432^2$. Since $123456m$ can be written as n^2 where $n = 15432 \in \mathbb{Z}$, it is a perfect square, and the statement is true. □

Note: To find $m = 1929$, notice that if $123456m = n^2$, then $\sqrt{123456m} = 8\sqrt{1929m}$ (after simplifying by prime factorization) must be an integer.

(f)

Proof. Consider the negation, $\forall k \in \mathbb{Z}, 8 \mid (4k^2 + 12k + 8)$. Notice that the open sentence is logically equivalent to $8 \mid (4k^2 + 12k)$. Let k be a natural number. Then, k is either even or odd.

Suppose that k is even and can be written as $k = 2n$. Then, $4k^2 = 16n^2 = 8(2n^2)$, so $8 \mid 4k^2$. Likewise, $12k = 24n = 8(3n)$, so $8 \mid 12k$. By DIC, $8 \mid (4k^2 + 12k)$.

Now, suppose that k is odd and can be written as $k = 2n + 1$. Then, $4k^2 + 12k = 4(4n^2 + 2n + 1) + 12(2n + 1) = 16n^2 + 40n + 16 = 8(2n^2 + 5n + 1)$, so $8 \mid (4k^2 + 12k)$.

Therefore, the negation is true, so the original statement is false. □

Recommended Problem 3.5. Prove or disprove each of the following statements involving nested quantifiers.

(a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.

Proof. Let n be an integer. If $n = 0$ or $n = \pm 1$, $n^3 - n = 0$ and all integers (including any k) divide zero.

If $n > 1$, we select $k = n + 1 > 2$. Factor: $n^3 - n = n(n-1)(n+1)$. Then, $n^3 - n = [n(n-1)](n+1)$, so $k \mid (n^3 - n)$.

If $n < 1$, first let $m = -n$ so $n^3 - n = (-m)^3 + m = -(m^3 - m)$. Now, select $k = m + 1 > 2$. Then, $n^3 - n = -m(m-1)(m+1)$, so $k \mid (n^3 - n)$.

Therefore, the statement is true. □

(b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.

Proof. We disprove by counter-example. Let $a = 1$. Then, $|b| < 1$, and the only such integer is 0. However, $0 \nmid 1$ since there is no integer k where $k \cdot 0 = 1$. Therefore, the statement is false. □

(c) There exists an integer n such that $m(n-3) < 1$ for every integer m.

Proof. Choose $n = 3$ and let m be an integer. Then, $m(n-3) = m(3-3) = 0 < 1$, as desired. Therefore, the statement is true. □

(d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$

Proof. Consider the negation $\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, -nm \geq 0$. Let n be a natural number.

We can choose an integer m, namely $m = -1$. Notice that because n is a natural number, $n > 0 \iff n(-1)(-1) > 0 \iff -nm > 0 \iff -nm \geq 0$.

Because the negation is true, the original statement is false. □

Recommended Problem 3.6. Prove that for all integers a and b, if $a \mid (2b+3)$ and $a \mid (3b+5)$, then $a \mid 13$.

Proof. Let a and b be arbitrary integers, and assume that $a \mid (2b + 3)$ and $a \mid (3b + 5)$.

Recall the divisibility of integer combinations: since $2b + 3$ and $3b + 5$ are integers, a must divide $n(2b + 3) + m(3b + 5)$ for all integers n and m. Specifically, let $n = -39$ and $m = 26$. Then, $n(2b + 3) + m(3b + 5) = -78b - 117 + 78b + 130 = 13$. Therefore, $a \mid 13$. □

Recommended Problem 3.7. Let a, b, c and d be positive integers. Prove that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof. Let a, b, c and d all be positive integers. Suppose $\frac{a}{b} < \frac{c}{d}$, which means $ad < bc$, because b

and d are positive. Now, adding ab and cd to both sides, respectively:

$$\begin{aligned} ad &< bc \\ ad + ab &< bc + ab \\ a(b+d) &< b(c+a) \\ \frac{a}{b} &< \frac{a+c}{b+d} \end{aligned} \qquad \begin{aligned} ad &< bc \\ ad + cd &< bc + cd \\ d(a+c) &< c(b+d) \\ \frac{a+c}{b+d} &< \frac{c}{d} \end{aligned}$$

Therefore, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

Recommended Problem 3.8. Prove that for all integers n, if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

Proof. Let n be an integer where $1 - n^2 > 0$. Since squares of integers are positive, $1 > n^2$. This is only true when $|n| < 1$, but the only such integer is 0. $3(0) - 2 = -2$, which is even. □

Recommended Problem 3.9. Let a and b be integers. Prove each of the following implications.

(a) If $ab = 4$, then $(a-b)^3 - 9(a-b) = 0$

Proof. Let a and b be integers with product 4.

Consider the possible values for a and b. 4's divisor pairs are $(\pm 1, \pm 4)$ and $(\pm 2, \pm 2)$. For all of these pairs, either $a = b$ or $a = b \pm 3$. Specifically:

- If $b = \pm 2$, then $a = b$
- If $b = 1$, then $a = 4 = b + 3$ (for $b = -1$, $a = -4 = b - 3$)
- If $b = 4$, then $a = 1 = b - 3$ (for $b = -4$, $a = -1 = b + 3$)

Notice that the conclusion factors to $(a-b)(a-b-3)(a-b+3) = 0$. This is true when $a = b$ or $a = b \pm 3$, which we just showed. □

(b) If a and b are positive, then $a^2(b+1) + b^2(a+1) \geq 4ab$

Proof. Let a and b be positive integers, i.e., at least 1.

If a and b are both at least 1, then $a + b \geq 2$, or $a + b - 2 \geq 0$. Likewise, ab is a positive integer, so $ab(a + b - 2) \geq 0$.

$$\begin{aligned} ab(a+b-2) &\geq 0 \\ a^2b + b^2a - 2ab &\geq 0 \end{aligned}$$

Recall that squares are non-negative:

$$\begin{aligned} (a-b)^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 - 2ab + b^2 + a^2b + b^2a - 2ab &\geq 0 \\ a^2 + a^2b + b^2 + b^2a &\geq 4ab \\ a^2(b+1) + b^2(a+1) &\geq 4ab \end{aligned}$$

□

Recommended Problem 3.10. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

Proof. Let a, b, c, and d be integers where $a \mid b$, $b \mid c$, and $c \mid d$.

Recall the transitivity of divisibility: for integers x, y, and z, if $x \mid y$ and $y \mid z$, then $x \mid z$.

Then, $a \mid b$ and $b \mid c$ implies $a \mid c$. Likewise, $a \mid c$ and $c \mid d$ implies $a \mid d$. □

Recommended Problem 3.11. Prove that the product of any four consecutive integers is one less than a perfect square.

Proof. The statement is equivalently expressed that for any integer k, $k(k+1)(k+2)(k+3) = r^2 - 1$ for some positive integer r.

Let k be an integer. The product $k(k+1)(k+2)(k+3)$ expands to $k^4 + 6k^3 + 11k^2 + 6k$. As a fourth-degree polynomial, its square root would be a quadratic.

Expanding algebraically, the square of a quadratic in x, $ax^2 + bx + c$, is $a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2$.

Notice that when $a = c = 1$ and $b = 3$, this formula becomes $x^4+6x^3+11x^2+6x+1$. The coefficients on x are precisely our original product (with a constant $+1$). Therefore, $x^4 + 6x^3 + 11x^2 + 6x = (x^2 + 3x + 1)^2 - 1$ for all real x.

We can now let $r = k^2 + 3k + 1$, which is a positive integer such that

$$\begin{aligned} r^2 - 1 &= (k^2 + 3k + 1)^2 - 1 \\ &= k^4 + 6k^3 + 11k^2 + 6k + 1 - 1 \\ &= k(k+1)(k+2)(k+3) \end{aligned}$$

and conclude that the statement is true. □

Recommended Problem 3.12. Let x, $y \in \mathbb{R}$. Prove that if $xy + 2x - 3y - 6 < 0$, then $x < 3$ or $y < -2$.

Proof. Let x and y be real solutions to $xy + 2x - 3y - 6 < 0$.

Notice that the inequality factors to $(x - 3)(y + 2) < 0$. This is true when x and y are non-zero and have opposite signs: either $x < 3$ and $y > -2$, or $x > 3$ and $y < -2$. Therefore, either $x < 3$ or $y < -2$. □

Recommended Problem 3.13. Is the following implication true for all integers a, b and c? Prove that your answer is correct.

$$a \mid b \text{ if and only if } ac \mid bc$$

Solution. The statement is false. Consider the counterexample $a = 2$, $b = 3$, and $c = 0$. Then, the backwards implication's hypothesis is true ($0 \mid 0$) but the conclusion is false ($2 \nmid 3$). □

Recommended Problem 3.14. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let n be an integer where 2 divides $n^4 - 3$. This means there is an integer k where $n^4 - 3 = 2k$. Notice that this means $n^4 - 3$ is even, so $n^4 = 2(k+1) + 1$ is odd. Even numbers raised to the fourth power remain even, so n must be odd. Therefore, $n = 2m + 1$ for some integer m.

Now, expand $n^2 + 3$:

$$\begin{aligned} n^2 + 3 &= (2m+1)^2 + 3 \\ &= 4m^2 + 4m + 1 + 3 \\ &= 4(m^2 + m + 1) \end{aligned}$$

Because $m^2 + m + 1$ is an integer, $4 \mid (n^2 + 3)$.

($\Leftarrow$) Let n be an integer where 4 divides $n^2 + 3$. This means there is an integer k where $n^2 + 3 = 4k$ or $n^2 = 4k - 3$, and

$$\begin{aligned} n^2 &= 4k - 3 \\ n^4 &= (4k-3)^2 \\ n^4 &= 16k^2 - 24k + 9 \\ n^4 - 3 &= 16k^2 - 24k + 6 \\ &= 2(8k^2 - 12k + 3) \end{aligned}$$

Because $8k^2 - 12k + 3$ is an integer, $2 \mid (n^4 - 3)$.

Therefore, since both expressions imply the other, $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$. □

Recommended Problem 3.15. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

Proof. Consider the contrapositive, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$.

Let x and y be non-zero integers. WLOG, take $x \leq y$.

Now, take cases of the signs of x and y:

- If $0 < x \leq y$, then $xy > 0$, since two positive numbers' product is a positive number.
- xy is also positive when $x \leq y < 0$, with two negative numbers.
- When $x < 0 < y$, i.e. the signs are opposite, $xy < 0$.

Since xy can never be 0 for any combination of non-zero integers, the contrapositive, and by extension, the original implication, is true. □

Recommended Problem 3.16. Prove that $\forall a, b \in \mathbb{Z}, [(a \mid b \wedge b \mid a) \iff a = \pm b]$.

Proof. Let a and b be integers. Suppose a divides b and vice versa. Equivalently, integers p and q exist such that $a = pb$ and $b = qa$. Substituting, $a = pb = p(qa) \iff 1 = pq \iff p = \frac{1}{q}$.

The only integers of the form $\frac{1}{k}$ with integer k are 1 and -1. Therefore, $p = \frac{1}{q}$ if and only if $p = \pm 1$, i.e., $a = \pm b$. □

Recommended Problem 3.17. Let a be an integer. Prove that $a^2 + 2a - 3$ is even if and only if a is odd.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let a be an odd integer, or, $a = 2k + 1$ for some integer k. Then,

$$\begin{aligned} 2a^2 + 2a - 3 & \\ &= (2k+1)^2 + 2(2k+1) - 3 \\ &= 4k^2 + 4k + 1 + 4k + 2 - 3 \\ &= 4k^2 + 8k - 2 \\ &= 2(2k^2 + 4k - 1) \end{aligned}$$

which is even, because $2k^2 + 4k - 1$ is an integer.

($\Leftarrow$) Consider the contrapositive, where even a implies odd $a^2 + 2a - 3$. Let a be an even integer, i.e., $a = 2k$ for some integer k. Then,

$$\begin{aligned} a^2 + 2a - 3 &= (2k)^2 + 2(2k) - 3 \\ &= 4k^2 - 4k - 3 \\ &= 2(2k^2 - 2k - 2) + 1 \end{aligned}$$

which is odd, because $2k^2 - 2k - 2$ is an integer. Since the contrapositive is true, the original implication is also true.

Therefore, since both implications hold, the statement is true. □

Recommended Problem 3.18. Prove or disprove each of the following for any integers x and y.

(a) If $2 \nmid xy$ then $2 \nmid x$ and $2 \nmid y$.

(b) If $2 \nmid y$ and $2 \nmid x$ then $2 \nmid xy$.

Proof. First, notice that if $2 \mid n$ for an integer n, then $n = 2k$ for some integer k. This is the definition of saying n is even. Therefore, $2 \nmid n$ is the same as saying n is not even, i.e., n is odd.

Let x and y be odd integers. Equivalently, $x = 2p + 1$ and $y = 2q + 1$ for some integers p and q. Substituting into xy, $(2p + 1)(2q + 1) = 2pq + 2p + 2q + 1 = 2(pq + p + q) + 1$. By definition, since $pq + p + q$ is an integer, xy is odd.

Therefore, x and y are odd if and only if xy is odd, so (a) and (b) are both true. □

(c) If $10 \nmid xy$ then $10 \nmid x$ and $10 \nmid y$.

Proof. Consider the contrapositive, "if $10 \mid x$ and $10 \mid y$ then $10 \mid xy$". Let x and y be integers where 10 divides both.

This means $x = 10n$ and $y = 10m$ for some integers n and m. Then, $xy = (10n)(10m) = 10(10nm)$, and since $10nm$ is an integer, $10 \mid xy$.

Since the contrapositive, the original implication is true. □

(d) If $10 \nmid x$ and $10 \nmid y$ then $10 \nmid xy$.

Proof. For a counterexample, let $x = 5$ and $y = 2$. $10 \nmid x$ and $10 \nmid y$ since $2 < 5 < 10$.

However, $xy = 10$ and $10 \mid 10$, so the statement is false. □

Recommended Problem 3.19. For every odd integer n, prove that there exists a unique integer m such that $n^2 = 8m + 1$.

Proof. Let n be an odd integer, i.e., $n = 2k + 1$ for some other integer k. Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. We must show that $8m = 4k^2 + 4k \iff 2m = k^2 + k$, or, $k^2 + k$ is even. Now, consider k's parity:

Suppose k is even. Then, $k = 2p$ for an integer p and $k^2 + k = 4p^2 + 2p = 2(2p^2 + p)$, which means that $k^2 + k$ is even.

Now, suppose k is odd. Then, $k = 2p+1$ for an integer p and $k^2+k = 4p^2+6p+2 = 2(2p^2+3p+1)$, which means that $k^2 + k$ is even.

Since k is either even or odd, $k^2 + k$ is even for all k.

Therefore, $m = \frac{k^2+k}{2}$ is an integer, but recall $k = \frac{n-1}{2}$, so:

$$m = \frac{k^2 + k}{2} = \frac{\left(\frac{n-1}{2}\right)^2 + \frac{n-1}{2}}{2} = \frac{\frac{(n-1)^2}{4} + \frac{n-1}{2}}{2} = \frac{(n-1)^2 + 2(n-1)}{8} = \frac{n^2 - 1}{8}$$

is the same integer, but $m = \frac{n^2-1}{8}$ if and only if $n^2 = 8m + 1$, so the statement is true. □

Recommended Problem 3.20. Prove the following statements.

(a) There is no smallest positive real number.

Proof. Suppose, for a contradiction, that there is a smallest positive real number n. Now, consider $\frac{n}{2}$.

This number is still real ($\mathbb{R}$ is closed under division). $\frac{n}{2}$ is positive because n and 2 are positive. Therefore, $\frac{n}{2}$ is a positive real number.

Clearly $\frac{n}{2} < n$, so the supposition must be false. Therefore, there is no smallest positive real number. □

(b) For every even integer n, n cannot be expressed as the sum of three odd integers.

Proof. We will prove by the contrapositive. Let r, s, and t be arbitrary integers so $2r+1$, $2s+1$, and $2t+1$ are odd.

Then, $r+s+t = 2r+2s+2t+3 = 2(r+s+t+1)+1$, so this sum is odd.

Therefore, the sum of three odd integers is always odd, and no even integer may be expressed as such a sum. □

(c) Let $a, b \in \mathbb{Z}$. If a is an even integer and b is an odd integer, then $4 \nmid (a^2+2b^2)$.

Proof. Let a and b be integers and suppose, for a contradiction, that the negation is true. Then, a is even, b is odd, and $4 \mid (a^2+2b^2)$.

Rewrite $a = 2n$ and $b = 2m+1$ with some integers n and m. Now, expand $a^2+2b^2 = (2n)^2+2(2m+1)^2 = 4n^2+8m^2+8m+2$.

We can extract a factor of four, and get $4 \mid (4(n^2+2m^2+2m)+2)$. Then, 4 must divide 2, which is a contradiction.

Therefore, the negation is false, so the original statement is true. □

(d) For every integer m with $2 \mid m$ and $4 \nmid m$, there are no integers x and y that satisfy $x^2+3y^2 = m$.

Proof. Those negations are ugly so we can consider the contrapositive:

$$\textit{If } x^2+3y^2 = m \textit{ has integer solutions in } x \textit{ and } y, \textit{ } m \textit{ is odd or } 4 \mid m.$$

Notice that $2 \nmid m \vee 4 \mid m \equiv 4 \mid m$.

Suppose integers x and y so $x^2+3y^2 = m$ exist. Break into cases for x and y's parities.

- If x and y are odd, they can respectively be expressed as $2p+1$ and $2q+1$ for integers p and q. Then, $m = (2p+1)^2+3(2q+1)^2 = 4p^2+4p+1+3(4q^2+4q+1)$. This simplifies to $4(p^2+3q^2+p+3q+1)$, so $m \mid 4$.
- If x and y are even, let $x = 2p$ and $y = 2q$. Then, $m = (2p)^2+3(2q)^2 = 4p^2+12q^2 = 4(p^2+3q^2)$, so $m \mid 4$.
- If x is odd and y is even, let $x = 2p+1$ and $y = 2q$. Then, $m = (2p+1)^2+3(2q)^2 = 4p^2+4p+1+3(4q^2) = 2(2p^2+2p+6q^2)+1$, so m is odd.
- If x is even and y is odd, let $x = 2p$ and $y = 2q+1$. Then, $m = (2p)^2+3(2q+q)^2 = 4p^2+3(4q^2+4q^2+1) = 2(2p^2+6q^2+6q+1)+1$, so m is odd.

Therefore, either 4 divides m or m is odd, so the contrapositive, and by extension the original statement, is true. □

(e) The sum of a rational number and an irrational number is irrational.

Proof. First, recall that rational numbers are those which can be expressed by $\frac{p}{q}$ for integers p and q.

Let x be a rational number and suppose for a contradiction that y is irrational such that $x+y$ is rational.

Then, $x = \frac{p}{q}$ for integers p and q. Also, $x + y = \frac{n}{m}$ for integers n and m. Substituting, $\frac{p}{q} + y = \frac{n}{m}$. Rearranging,

$$\frac{p}{q} + y = \frac{n}{m} \iff p + yq = \frac{qn}{m} \iff y = \frac{qn - mp}{qm}$$

but if y equals the ratio of two integers ($qn - mp$ and qm), by definition, y is rational.

Therefore, by contradiction, the sum of a rational number and an irrational number is irrational. □

(f) Let x be a non-zero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.

Proof. Let x be a non-zero real such that $x + \frac{1}{x} < 2$. Then,

$$\begin{aligned} x + \frac{1}{x} &< 2 \\ x + \frac{1}{x} - 2 &< 0 \\ \frac{x^2 + 1 - 2x}{x} &< 0 \\ \frac{(x-1)^2}{x} &< 0 \end{aligned}$$

Because $(x-1)^2$ is a square, so is always non-negative, $\frac{1}{x} < 0$, which is true if and only if $x < 0$. □

3.3 Challenges

Challenge 3.1. Let n be an integer. Prove that if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

Proof. Let n be an integer such that $2 \mid n$ and $3 \mid n$. Then, there exist integers $2p = n$ and $3q = n$. Equivalently, $6p = 3n$ and $6q = 2n$. Subtracting, $n = 6(p - q)$, and since $p - q$ is an integer, $6 \mid n$. □

Challenge 3.2. Let $a, b, c \in \mathbb{R}$. Prove that if $a^2 + b^2 + c^2 = 1$, then $-1/2 \leq ab + bc + ca \leq 1$.

Proof. Let a, b, and c be real numbers. Recall that squares of reals are non-negative. Then, notice that we can create $2ab$-type terms in squares of binomials:

$$\begin{aligned} 0 &\leq (a-b)^2 + (b-c)^2 + (c-a)^2 \\ 0 &\leq 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \\ 0 &\leq (a^2 + b^2 + c^2) - (ab + bc + ca) \\ ab + bc + ca &\leq 1 \end{aligned}$$

Likewise, we can create these terms in the square of a trinomial:

$$\begin{aligned} 0 &\leq (a+b+c)^2 \\ 0 &\leq a^2+b^2+c^2+2ab+2bc+2ca \\ -(a^2+b^2+c^2) &\leq 2(ab+bc+ca) \\ -\frac{1}{2} &\leq ab+bc+ca \end{aligned}$$

Therefore, combining these inequalities, $-\frac{1}{2} \leq ab+bc+ca \leq 1$, as desired. □

Challenge 3.3. Show that if p and p^2+2 are prime, then p^3+2 is also prime.

Challenge 3.4. Express the following statement in symbolic form and prove that it is true.

> There exists a real number L such that for every positive real number ϵ, there exists a positive real number δ such that for all real numbers x, if $|x| < \delta$, then $|3x - L| < \epsilon$.

Proof. In symbolic form, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$:

$$\exists L \in \mathbb{R}, \forall \epsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x| < \delta \implies |3x - L| < \epsilon$$

We propose $L = 0$. Let $\epsilon > 0$. Select $\delta = \frac{\epsilon}{3}$. Now, suppose that $|x| < \delta$. Then, $|3x - L| = |3x| = 3|x| < 3\delta = 3\frac{\epsilon}{3} = \epsilon$, as desired. □

Challenge 3.5. Prove that there are no positive integers a and b such that $b^4 + b + 1 = a^4$.

Challenge 3.6. Prove that the length of at least one side of a right-angled triangle with integer side lengths must be divisible by 3.

Chapter 4

Mathematical Induction

4.1 Warm-Up Exercises

Warm-Up Exercise 4.1. Evaluate $\sum_{i=3}^{8} 2^i$ and $\prod_{j=1}^{5} \frac{j}{3}$.

Solution. Simply expand along the sum/product:

$$\sum_{i=3}^{8} 2^i = 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 = 8 + 16 + 32 + 64 + 128 + 256 = 504$$

and

$$\prod_{j=1}^{5} \frac{j}{3} = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot \frac{4}{3} \cdot \frac{5}{3} = \frac{120}{243} = \frac{40}{81}$$ □

Warm-Up Exercise 4.2. Let x be a real number. Using the Binomial Theorem, expand $\left(x - \frac{1}{x}\right)^7$.

Solution. Recall the Binomial Theorem, that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Now, substitute $a = x$

and $b = -\frac{1}{x}$.

$$\begin{aligned}\left(x - \frac{1}{x}\right)^7 &= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} \left(-\frac{1}{x}\right)^k \\ &= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} x^{-k} (-1)^k \\ &= \sum_{k=0}^{7} \binom{7}{k} (-1)^k x^{7-2k} \\ &= x^7 - 7x^{7-2} + 21x^{7-4} - 35x^{7-6} + 35x^{7-8} - 21x^{7-10} + 7x^{7-12} - x^{7-14} \\ &= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}\end{aligned}$$

□

4.2 Recommended Problems

Recommended Problem 4.1. Prove the following statements by induction.

(a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n}(2i-1) = n^2$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n}(2i-1) = n^2$ on n.

(Base Case) When $n = 1$, the left-hand side is

$$\begin{aligned}\sum_{i=1}^{1}(2i-1) &= 2(1) - 1 \\ &= 1 \\ &= 1^2\end{aligned}$$

which is the right-hand side, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Then, we take the left-hand side of $P(k+1)$

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= (2(k+1)-1) + \sum_{i=1}^{k}(2i-1) \\ &= (2k+1) + k^2 && \text{by inductive hypothesis} \\ &= (k+1)^2\end{aligned}$$

as desired to show that if $P(k)$ holds, then $P(k+1)$ holds.

Therefore, by induction, $P(n)$ holds for all n. □

(b) For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ where r is any real number such that $r \neq 1$.

Proof. Let r be an arbitrary real other than 1. We will induct the statement $P(n) \equiv \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ on n.

(Base Case) For $n = 1$, substitute into the LHS and expand the summation:

$$\sum_{i=0}^{1} r^i = r^0 + r^1 = 1 + r = (1+r)\frac{1-r}{1-r} = \frac{1-r^2}{1-r}$$

This is precisely the RHS of the equality, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Again, expand the summation but for the LHS of $P(k+1)$:

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= r^{k+1} + \sum_{i=0}^{k} r^i \\
&= r^{k+1} + \frac{1-r^{k+1}}{1-r} && \text{by inductive hypothesis} \\
&= \frac{(r^{k+1})(1-r) + 1 - r^{k+1}}{1-r} \\
&= \frac{r^{k+1} - r^{k+2} + 1 - r^{k+1}}{1-r} \\
&= \frac{1-r^{k+2}}{1-r}
\end{aligned}$$

which is the other side of the equality. We have proved that if $P(n)$ holds, then $P(n+1)$ holds. Therefore, by induction, $P(n)$ holds for all natural n. □

(c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$ on n.

First, verify the base case, $P(1)$. Then, we let $n = 1$ and have

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = 1 - \frac{1}{2!}$$

Expanding the summation, we can show that $P(1)$ holds:

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$$

Now, suppose $P(k)$ is true for some k, and consider $P(k+1)$:

$$\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+2)!}$$

Like above, we take out a term of the summation and simplify, so we have

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \frac{k+1}{(k+2)!} + \sum_{i=1}^{k} \frac{i}{(i+1)!} \\ &= \frac{k+1}{(k+2)!} + 1 - \frac{1}{(k+1)!} && \text{by inductive hypothesis} \\ &= 1 + \frac{(k+1)-(k+2)}{(k+2)!} \\ &= 1 - \frac{1}{(k+2)!}\end{aligned}$$

as required. We have proven $P(1)$ and that $P(k)$ implies $P(k+1)$, so, by induction, $P(n)$ is true for all natural n. □

(d) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Proof. For induction on n, let $P(n) \equiv \sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Verify the base case $P(1)$:

$$\sum_{i=1}^{1} \frac{i}{2^i} = \frac{1}{2} = 2 - \frac{3}{2} = 2 - \frac{1+2}{2^1}$$

Suppose that $P(k)$ holds for some k, and consider $P(k+1)$. Now,

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{i}{2^i} &= \frac{k+1}{2^{k+1}} + \sum_{i=1}^{n} \frac{i}{2^i} \\ &= \frac{k+1}{2^{k+1}} + 2 - \frac{k+2}{2^k} && \text{by inductive hypothesis} \\ &= 2 + \frac{k+1-2(k+2)}{2^{k+1}} \\ &= 2 - \frac{k+3}{2^{k+1}}\end{aligned}$$

as required. Because $P(1)$ holds and $P(k)$ implies $P(k+1)$, by induction, $P(n)$ holds for all n. □

(e) For all $n \in \mathbb{N}$, where $n \geq 4$, $n! > n^2$.

Proof. We will prove by induction on n. Let $P(n)$ be the statement $n! > n^2$.

To verify the base case $P(4)$, notice that $4! = 24$, that $4^2 = 16$, and that $24 > 16$.

Now, suppose that $P(k)$ is true for some $k \geq 4$. We must show that $P(k+1)$ holds, i.e., $(k+1)! > (k+1)^2$.

First, notice that $x^2 > x + 1$ for all $x \geq 4$. Then, we can state the inductive hypothesis as $k! > k + 1$. Multiplying both sides by $k + 1$ gives $(k+1)! > (k+1)^2$, as desired.

Therefore, by induction, $n! > n^2$ for all $n \geq 4$. □

(f) For all $n \in \mathbb{N}$, $4^n - 1$ is divisible by 3.

Proof. Induct the statement "$4^n - 1$ is divisible by 3" on n.

For the base case, let $n = 1$ so $4^1 - 1 = 3$ and 3 is obviously divisible by 3.

Now, suppose that $4^k - 1$ is divisible by 3 for some natural number k. By definition, there exists an integer a where $4^k - 1 = 3a$.

Consider when $n = k + 1$. Rearranging, $4^{k+1} - 1 = (4^{k+1} - 4) + 3 = 4(4^k - 1) + 3$. By our inductive hypothesis, this is equal to $4(3a) + 3 = 3(4a + 1)$. Then, since $4^{k+1} - 1$ can be written as $3b$ for some integer b (namely, $b = 4a + 1$), it is by definition divisible by 3.

Therefore, by induction, $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$. □

Recommended Problem 4.2. Let x be a real number. Find the coefficient of x^{19} in the expansion of $(2x^3 - 3x)^9$.

Solution. Recall the Binomial Theorem, $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Let $a = 2x^3$, $b = -3x$, and $n = 9$. Then, we have $(2x^3 - 3x)^9 = \sum_{k=0}^{9} \binom{9}{k} 2^{9-k}(-3)^k x^{27-2k}$. We only care about when the exponent on x is 19, i.e., $27 - 2k = 19 \implies k = 4$. On this term of the summation, we have $\binom{9}{4} 2^5 (-3)^4 x^{19}$.

The coefficient is $\binom{9}{4} 2^5 (-3)^4 = 126 \cdot 32 \cdot 81 = 326592$. □

Recommended Problem 4.3. Let n be a non-negative integer. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. We will induct the statement $P(n) \equiv \sum_{k=0}^{n} \binom{n}{k} = 2^n$ on $n \geq 0$.

For the base case, $P(0)$, we have

$$\sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0.$$

Now, suppose $P(m)$ is true for some $m \geq 0$. Consider the summation in $P(m+1)$:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m+1}{k} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) && \text{by Pascal's identity} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m}{k} + \sum_{k=0}^{m} \binom{m}{k-1} \\
&= 1 + 2^k + \sum_{k=0}^{m} \binom{m}{k-1} && \text{by inductive hypothesis}
\end{aligned}$$

Recall that negative binomial coefficients are undefined, so we can change the variable in the summation with $j = k + 1$ and ignore the $k = 0$ term. Add and subtract a $\binom{m}{m}$ term to round out the summation and apply the IH once more:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} \\
&= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} + \binom{m}{m} - \binom{m}{m} \\
&= 1 + 2^k + \sum_{j=0}^{m} \binom{m}{j} - 1 \\
&= 1 + 2^k + 2^k - 1 && \text{by inductive hypothesis} \\
&= 2^{k+1}
\end{aligned}$$

which is what we wanted to show that $P(m+1)$ is true.

Therefore, by induction, $P(n)$ is true for all non-negative integer n. □

Recommended Problem 4.4. Let n be a non-negative integer. Prove by induction on k that $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$ for all non-negative integers k.

Proof. Let $n \geq 0$ be an integer, and let $P(k)$ be the statement $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$. We will induct $P(k)$ on k.

For the base case, let $k = 0$. Then, $P(k)$ reads $\sum_{j=0}^{0} \binom{n+j}{j} = \binom{n+1}{0}$. The summation only has one term, so we have $\binom{n}{0} = \binom{n+1}{0}$ which is true for all n (since $\binom{a}{0} = 1$ for all a).

Now, suppose that $P(s)$ holds for some non-negative integer s.

This means that $\sum_{j=0}^{s} \binom{n+j}{j} = \binom{n+s+1}{s}$. Now, consider the left-hand side of $P(s+1)$:

$$\begin{aligned}
\sum_{j=0}^{s+1} \binom{n+j}{j} &= \binom{n+s+1}{s+1} + \sum_{j=0}^{s} \binom{n+j}{j} \\
&= \binom{n+s+1}{s+1} + \binom{n+s+1}{s} && \text{by inductive hypothesis} \\
&= \binom{n+s+2}{s+1} && \text{by Pascal's identity}
\end{aligned}$$

which is exactly the right-hand side. Since $P(n)$ is true for $n = 0$ and $P(s)$ implies $P(s+1)$, it holds for all $n \geq 0$ by induction. □

Recommended Problem 4.5. The sequence $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 8$, $x_2 = 32$, and $x_i = 2x_{i-1} + 3x_{i-2}$ for all integers $i \geq 3$. Prove that for all $n \in \mathbb{N}$, $x_n =$

$2 \times (-1)^n + 10 \times 3^{n-1}$.

Proof. We will strongly induct the statement $P(n)$, $x_n = 2(-1)^n + 10(3)^{n-1}$, on n.

For a base case, let $n = 1$. Then, $2(-1)^1 + 10(3)^0 = -2 + 10 = 8$, which is the defined value of x_1. For another, let $n = 2$. Then, $2(-1)^2 + 10(3)^1 = 2 + 30 = 32$, which is the defined value of x_2. Therefore, $P(1)$ and $P(2)$ hold.

Now, for some $m \geq 3$, suppose $P(n)$ holds for all $n < m$. Specifically, $P(m-1)$ and $P(m-2)$ hold.

Consider the definition of x_m:

$$\begin{aligned} x_m &= 2x_{m-1} + 3x_{m-2} \\ &= 2\left(2(-1)^{m-1} + 10(3)^{m-2}\right) + 3\left(2(-1)^{m-2} + 10(3)^{m-3}\right) \\ &= 4(-1)^{m-1} + 20(3)^{m-2} + 6(-1)^{m-2} + 30(3)^{m-3} \\ &= 4(-1)(-1)^{m-2} + 6(-1)^{m-2} + 20(3)(3)^{m-3} + 30(3)^{m-3} \\ &= 2(-1)^{m-2} + 90(3)^{m-3} \\ &= 2(-1)^2(-1)^{m-2} + 10(3)^2(3)^{m-3} \\ &= 2(-1)^m + 10(3)^{m-1} \end{aligned}$$

which is precisely $P(m)$.

Therefore, by strong induction, $P(n)$ is true for all n. □

Recommended Problem 4.6. The sequence $t_1, t_2, t_3, \ldots$ is defined recursively by $t_1 = 2$ and $t_n = 2t_{n-1} + n$ for all integers $n > 1$. Prove that for all $n \in \mathbb{N}$, $t_n = 5 \times 2^{n-1} - 2 - n$.

Proof. Let $P(n)$ be the statement $t_n = 5 \times 2^{n-1} - 2 - n$. We will induct $P(n)$ on n.

We first verify base cases: $n = 1$, hypothesized as $t_1 = 5(2)^0 - 2 - 1 = 2$, which matches the defined value; and $n = 2$, for which t_2 is defined as $2(2) + 2 = 6$ and we hypothesize $t_2 = 5(2)^1 - 2 - 2 = 6$.

Now, let m be an integer above 2 and suppose that $P(m-1)$ holds. Consider the definition of t_m:

$$\begin{aligned} t_m &= 2t_{m-1} + m \\ &= 2\left(5(2)^{m-2} - 2 - (m-1)\right) + m && \text{by inductive hypothesis} \\ &= 2\left(5(2)^{m-2} - m - 1\right) + m \\ &= 5(2)^{m-1} - 2m - 2 + m \\ &= 5(2)^{m-1} - 2 - m \end{aligned}$$

This is exactly $P(m)$, so $P(m-1)$ implies $P(m)$.

Therefore, by induction, $P(n)$ is true for all natural n. □

Recommended Problem 4.7. The Fibonacci sequence is defined as the sequence $f_1, f_2, f_3, \ldots$

where $f_1 = 1$, $f_2 = 1$ and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 3$. Use induction to prove the following statements:

(a) For $n \geq 2$, $f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$.

Proof. We will induct the statement $P(n)$, $\sum_{i=1}^{n-1} f_i = f_{n+1} - 1$ on n.

To verify the base case, $n = 2$, substitute and notice $f_1 = 1 = 2 - 1 = f_3 - 1$.

Now, let $m > 2$ and suppose $P(m)$ holds. Then,

$$\begin{aligned} \sum_{i=1}^{m-1} f_i &= f_{m+1} - 1 \\ f_m + \sum_{i=1}^{m-1} f_i &= f_m + f_{m+1} - 1 \\ \sum_{i=1}^{m} f_i &= f_{m+2} - 1 \end{aligned}$$

which is $P(m+1)$.

Therefore, by induction, $P(n)$ is true for all $n \geq 2$. □

(b) Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, $f_n = \dfrac{a^n - b^n}{\sqrt{5}}$.

Proof. Let $P(n)$ be the statement $f_n = \frac{a^n - b^n}{\sqrt{5}}$. We will strongly induct $P(n)$ on n.

For the base cases, start with $n = 1$. f_1 is defined to be 1, and $\frac{a-b}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$. Likewise, for $n = 2$, f_2 is defined as 1, and $\frac{a^2-b^2}{\sqrt{5}} = \frac{a-b}{\sqrt{5}}(a+b) = (1)(1) = 1$.

For our inductive step, first notice that a and b are the roots of $x^2 - x - 1 = 0$. Let x be either root.

Notice that for any $n \geq 2$, we have

$$\begin{aligned} 0 &= x^2 - x - 1 \\ 0 &= x^{n-2}(x^2 - x - 1) \\ 0 &= x^n - x^{n-1} - x^{n-2} \\ x^n &= x^{n-1} + x^{n-2} \end{aligned}$$

Therefore, $a^n = a^{n-1} + a^{n-2}$ and $b^n = b^{n-1} + b^{n-2}$ for any $n \geq 2$.

Now, let $m \geq 2$ and suppose $P(m-1)$ and $P(m-2)$ hold. Then, f_m is defined by:

$$\begin{aligned} f_m &= f_{m-1} + f_{m-2} \\ &= \frac{a^{m-1} - b^{m-1}}{\sqrt{5}} + \frac{a^{m-2} - b^{m-2}}{\sqrt{5}} \\ &= \frac{(a^{m-1} + a^{m-2}) - (b^{m-1} + b^{m-2})}{\sqrt{5}} \\ &= \frac{a^m - b^m}{\sqrt{5}} \end{aligned}$$

Therefore, by strong induction, $P(n)$ holds for all n. □

Recommended Problem 4.8. Each of the following "proofs" by induction incorrectly "proved" a statement that is actually false. State what is wrong with each proof.

(a) The proof does not consider the given definition $x_2 = 20$, and $3(5)^1 = 15 \neq 20$. Note that the recursive definition *only* applies to x_i for $i \geq 3$.

(b) The proof erroneously assumes that $n = 2$ always falls within the inductive hypothesis. However, when proving the case $n = 2$ with strong induction, the only given is $n = 1$.

Recommended Problem 4.9. In a strange country, there are only 4 cent and 7 cent coins. Prove that any integer amount of currency greater than 17 cents can always be formed.

Proof. Let $P(x)$ be the statement that there exist non-negative integer a and b where $x = 4a + 7b$. We will strongly induct on $x > 17$.

Verify a few base cases:
For $P(18)$ (where $18 = 4(4) + 2$), let $a = 1$ and $b = 2$, so $4(1) + 7(2) = 18$.
For $P(19)$ (where $19 = 4(4) + 3$), let $a = 3$ and $b = 1$, so $4(3) + 7(1) = 19$.
For $P(20)$ (where $20 = 4(5) + 0$), let $a = 5$ and $b = 0$, so $4(5) + 7(0) = 20$.
For $P(21)$ (where $21 = 4(5) + 1$), let $a = 0$ and $b = 3$, so $4(0) + 7(3) = 21$.

Now, suppose for some $n > 21$, $P(m)$ holds for all $m < n$. Specifically, $P(n - 4)$ holds. That is, $n - 4 = 4a_0 + 7b_0$ for some a_0 and b_0. Equivalently, $n = 4(a_0 + 1) + 7b_0$. If we let $a = a_0 + 1$ and $b = b_0$, it follows that $P(n)$ holds.

Therefore, by strong induction, $P(x)$ is true for all $x > 17$. □

4.3 Challenges

Challenge 4.1. Prove that for every positive integer, there exists a unique way to write the integer as the sum of distinct non-consecutive Fibonacci numbers.

Proof. Let f_i denote the ith Fibonacci number, i.e., $f_1 = 0$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$. Note that we proceed without loss of generality with increasing lists of Fibonacci numbers.

We begin by proving inductively that $f_n > f_{k_1} + \cdots + f_{k_m}$ where $k_1 < \cdots < k_m < n$ and $k_1 + 1 \neq k_2$, $k_2 + 1 \neq k_3$, etc. That is, the k_i are increasing, and non-consecutive. For the cases $n = 0$ and $n = 1$, no such sums can exist. When $n = 2$, the only such sum is f_0, and $0 < f_2 = 1$.

Suppose that $n \geq 2$ and $f_{n-2} > f_{r_1} + \cdots + f_{r_s}$ with the r_i increasing and non-consecutive. Then, since $k_m < n$, $k_m \leq n - 1$ and we have

$$\begin{aligned} f_{k_1} + \cdots + f_{k_{m-1}} + f_{k_m} &\leq f_{k_1} + \cdots + f_{k_{m-1}} + f_{n-1} \\ &< f_{n-2} + f_{n-1} && \text{by inductive hypothesis} \\ &= f_n \end{aligned} \tag{4.1}$$

Now, let $P(n)$ be the statement that all positive integers $x < f_n$, $x = \sum_{i=1}^{m} f_{k_i}$ for unique, increasing, non-consecutive k_i.

For the base cases $P(1)$, $P(2)$, and $P(3)$ there are no positive integers $x < 0$ or $x < 1$. For the base case $P(4)$, the only positive integer less than $f_4 = 2$ is $x = 1$. Trivially, we can uniquely write $f_1 + f_3 = 0 + 1 = 1$.

For the inductive step, suppose that $P(n)$ holds for some $n \geq 4$. Let $f_n \leq x < f_{n+1}$.

If x is f_n, then we may write $x = f_1 + f_n$. That is, $x = \sum_{i=1}^{2} f_{k_i}$ with increasing, non-consecutive $k_1 = 1$ and $k_2 = n$.

Otherwise, write $x = d + f_n$ where $0 < d < f_{n+1} - f_n = f_{n-1}$. We now have, $d < f_{n-1} < f_n$ with positive integer d. By the inductive hypothesis, $d = \sum_{i=1}^{m} f_{k_i}$ for unique, increasing, non-consecutive k_i. Then, since $d < f_{n-1} < f_n$, none of the k_is can be n or $n-1$. Finally, let $k_{m+1} = n$ so that $x = \sum_{i=1}^{m+1} f_{k_i}$ has increasing, non-consecutive f_{k_i}.

Now, we show that the integers k_i are unique. Suppose $x = \sum_{i=1}^{m+1} f_{k_i} = \sum_{i=1}^{m+1} f_{\ell_i}$. We show that $k_i = \ell_i$ for all i.

Since both sums are increasing, the largest k_{m+1} is n. If $f_{\ell_{m+1}} > f_n$, then the sum is greater than f_{n+1}. But $x < f_{n+1}$, so this is a contradiction. If $f_{\ell_{m+1}} < f_n$, then by eq. (4.1), the sum is less than f_n. But $x \geq f_n$, so this is again a contradiction. Thus, $\ell_{m+1} = n = k_{m+1}$.

Then, $\sum_{i=1}^{m} f_{\ell_i} = x - f_n = d$. However, the inductive hypothesis gives that $\sum_{i=1}^{m} k_i$ is a unique representation of d. It follows that the remaining $\ell_i = k_i$ for all $i \leq m$.

Therefore, since we have proven $P(n+1)$, by induction, $P(n)$ holds for all n. □

Challenge 4.2. Find a formula for the minimum steps required to solve the Tower of Hanoi puzzle with three pegs with n rings. Prove that your answer is correct.

Chapter 5

Sets

5.1 Warm-Up Exercises

Warm-Up Exercise 5.1. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 9\}$, and $B = \{4, 5, 6, 7\}$.

(a) Calculate the following:

i. $A \cup B = \{2, 4, 5, 6, 7, 9\}$

ii. $A \cap B = \{4, 6\}$

iii. $\overline{A} = \mathcal{U} - A = \{1, 3, 5, 7, 8\}$

iv. $\overline{B} = \mathcal{U} - B = \{1, 2, 3, 8, 9\}$

v. $A - B = \{2, 9\}$

vi. $B - A = \{5, 7\}$

(b) Are A and B disjoint? *No, since 4 is in both sets.*

(c) Give a set C such that $C \subseteq B$. *Let $C = B$.*

(d) Give a set D such that $D \subsetneq A$. *Let $D = \{2\}$.*

Warm-Up Exercise 5.2. Suppose S and T are two sets. Prove that if $S \cap T = S$, then $S \subseteq T$. Is the converse true?

Proof. Let S and T be arbitrary sets such that their intersection is S. We must show that any element of S is an element of T.

Consider an element s in S. But S is equal to $S \cap T$. Elements of an intersection are elements of the original sets, so $s \in T$, as desired.

For the converse, consider another two sets, S_1 and T_1, where $S_1 \subseteq T_1$. This means that all elements of S_1 are elements of T_1, that is, all elements of S_1 are elements of both S_1 and T_1. But this is just the definition of the intersection of S_1 and T_1. Therefore, the converse is also true. □

Warm-Up Exercise 5.3. Give an example of three sets A, B, and C such that $B \neq C$ and $B - A = C - A$.

Solution. Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$. Then, $B - A = \{2\}$ and $C - A = \{2\}$. □

5.2 Recommended Problems

Recommended Problem 5.1. Let A be a subset of the universe $\mathcal{U}$. Prove that $A \cup \overline{A} = \mathcal{U}$.

Proof. Recall that the complement of a set $\overline{S}$ with respect to a universe $\mathcal{U}$ is defined as the set $\{x \in \mathcal{U} : \neg(x \in S)\}$. Recall also that the union of two sets X and Y, again with universe $\mathcal{U}$, is defined as the set $X \cup Y = \{x \in \mathcal{U} : x \in X \vee x \in Y\}$.

Then, $A \cup \overline{A} = \{x \in \mathcal{U} : x \in A \vee \neg(x \in A)\}$. The disjunction of any logical statement with its negation is a tautology, so this property is true for all elements of $\mathcal{U}$. Therefore, the resulting set is simply $\mathcal{U}$. □

Recommended Problem 5.2. Let S and T be two sets which are subsets of the universe $\mathcal{U}$. Prove that

$$(S \cup T) - (S \cap T) = (S - T) \cup (T - S).$$

Proof. Let S and T be arbitrary subsets of $\mathcal{U}$, and x be an arbitrary element of $\mathcal{U}$ such that it is an element of the left-hand side. We prove by showing that the left and right-hand sides are subsets of another, that is, the following universally quantified biconditional holds:

$$\forall x \in \mathcal{U}, x \in (S \cup T) - (S \cap T) \iff x \in (S - T) \cup (T - S)$$

This can be done by rewriting both sides in set-builder notation and applying logical equivalencies.

$$\begin{aligned}(S \cup T) - (S \cap T) &= \{x \in \mathcal{U} : (x \in S \cup T) \wedge (x \notin S \cap T)\} \\ &= \{x \in \mathcal{U} : (x \in S \vee x \in T) \wedge \neg(x \in S \wedge x \in T)\} \\ &= \{x \in \mathcal{U} : (x \in S \vee x \in T) \wedge (\neg(x \in S) \vee \neg(x \in T))\}\end{aligned}$$

Now, distributing, we have the property:

$$(x \in S \wedge x \notin S) \vee (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S) \vee (x \in T \wedge x \notin T)$$

which can be equivalently expressed by removing falsities:

$$(x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S).$$

Now, we can apply the definitions of unions and complements in reverse:

$$\begin{aligned}(S \cup T) - (S \cap T) &= \{x \in \mathcal{U} : (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)\} \\ &= \{x \in \mathcal{U} : (x \in S \wedge x \notin T)\} \cup \{x \in \mathcal{U} : (x \in T \wedge x \notin S)\} \\ &= (S - T) \cup (T - S)\end{aligned}$$

□

Recommended Problem 5.3. Let $A = \{n \in \mathbb{Z} : 2 \mid n\}$ and $B = \{n \in \mathbb{Z} : 4 \mid n\}$. Let $n \in \mathbb{Z}$. Prove that $n \in (A - B)$ if and only if $n = 2k$ for some odd integer k.

Proof. We prove the biconditional by proving both implications.

($\Rightarrow$) Let n be an arbitrary integer element of $A - B$, i.e., $n \in A$ but $n \notin B$. Then, the defining property of A holds but that of B does not. Therefore, $2 \mid n$ but $4 \nmid n$.

Since $2 \mid n$, it may be written as $n = 2q$ for some integer q.

If q is even, then $n = 2(2p)$ for some integer p. That means $n = 4p$, so $n \mid 4$, which is a contradiction. Therefore, q must be odd, and n may be written as $n = 2k$ for an odd integer $k = q$.

($\Leftarrow$) Let n be an arbitrary integer such that $n = 2k$ for some odd integer k. It immediately follows that $2 \mid n$ and $n \in A$.

Also, since k is odd, $n = 2(2d+1) = 4\left(d + \frac{1}{2}\right)$ for another integer d. $d + \frac{1}{2}$ will never be an integer, so $4 \nmid n$, which means $n \notin B$.

However, $n \in A$ and $n \notin B$ is precisely the definition of $n \in (A - B)$.

Therefore, since both implications hold, the statement is true. □

Recommended Problem 5.4. Prove that there exist sets A, B, and C such that $A \cup B = A \cup C$ and $B \neq C$.

Proof. Let $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. Clearly, $B \neq C$.

We have $A \cup B = \{1, 2\}$ and $A \cup C = \{1, 2\}$, which are equal. □

Recommended Problem 5.5. Prove or disprove. If A, B, and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup C$.

Solution. Let A, B, and C be arbitrary sets. We disprove by showing $(A \cap B) \cup C$ is not a subset of $A \cap (B \cup C)$.

Let x be an element of C that is not an element of A. Then, it is clearly an element of $(A \cap B) \cup C$, since it is an element of C and all elements of either set in a union are elements of the union.

However, it is not an element of $A \cap (B \cup C)$. Since it is an intersection, all such elements are elements of A, which x is not.

Therefore, $(A \cap B) \cup C \nsubseteq A \cap (B \cup C)$. Set equality is defined by bidirectional subsets, so the sets cannot be equal. □

Recommended Problem 5.6. Prove there is a unique set T such that for every set S, $S \cup T = S$.

Proof. We suppose that $T = \varnothing$, that is, T is the set with no elements, and prove it.

(Existence) Since there are no elements in T, it may be written as $T = \{x : P\}$ where P is a false logical statement.

Now, the union $S \cup T$ is $\{x : x \in S \vee P\}$. but a statement disjoined with false gives itself, so we have $\{x : x \in S\}$, which is just S.

(Uniqueness) Let A and B be empty sets.

Then, $\forall x \in \mathcal{U}, x \in A \implies x \in B$ is vacuously true, since the hypothesis is always false by definition. Therefore, $A \subseteq B$.

Likewise, $\forall x \in \mathcal{U}, x \in B \implies x \in A$ is vacuously true. Therefore, $B \subseteq A$.

Since both A and B are mutual subsets, $A = B$, and the empty set is unique. □

5.3 Challenges

Challenge 5.1. The *symmetric difference* of two sets A and B, denoted $A \triangle B$, is defined as

$$A \triangle B = (A - B) \cup (B - A).$$

Prove that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Proof. We will prove using logical equivalences.

Consider the left-hand side. By the given definition,

$$\begin{aligned}(A \triangle B) \triangle C &= ((A - B) \cup (B - A)) \triangle C \\ &= (((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A)))\end{aligned}$$

which is a mess, so we re-express as a logical expression in set-builder notation. That is, $\{x : P(x)\}$ for some open sentence $P(x)$. For convenience, let $a \equiv x \in A$, $b \equiv x \in B$, and $c \equiv x \in C$.

$$\begin{aligned}P(x) &\equiv (((a \wedge \neg b) \vee (b \wedge \neg a)) \wedge \neg c) \vee (c \wedge \neg((a \wedge \neg b) \vee (b \wedge \neg a))) \\ &\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge \neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a)) \\ &\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge (\neg a \vee b) \wedge (\neg b \vee a))\end{aligned}$$

We now digress from this (also enormous) expression to simplify the last term. Recall in RP 5.2, we proved $(X \vee Y) \wedge (\neg X \vee \neg Y) \equiv (X \wedge Y) \vee (\neg X \wedge \neg Y)$. We may now apply this equivalence with $X \equiv \neg a$ and $Y \equiv b$.

$$\begin{aligned}P(x) &\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge ((\neg a \wedge b) \vee (\neg b \wedge a))) \\ &\equiv (a \wedge \neg b \wedge \neg c) \vee (b \wedge \neg a \wedge \neg c) \vee (c \wedge \neg a \wedge b) \vee (c \wedge \neg b \wedge a) \\ &\equiv (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)\end{aligned}$$

Now, consider the right-hand side. By the given definition,

$$\begin{aligned}A \triangle (B \triangle C) &= A \triangle ((B - C) \cup (C - B)) \\ &= (A - ((B - C) \cup (C - B))) \cup (((B - C) \cup (C - B)) - A)\end{aligned}$$

which we may express as $\{x : Q(x)\}$ for some open sentence $Q(x)$.

$$\begin{aligned} Q(x) &\equiv (a \wedge \neg((b \wedge \neg c) \vee (c \wedge \neg b))) \vee (((b \wedge \neg c) \vee (c \wedge \neg b)) \wedge \neg a) \\ &\equiv (a \wedge (\neg(b \wedge \neg c) \wedge \neg(c \wedge \neg b))) \vee ((b \wedge \neg c \wedge \neg a) \vee (c \wedge \neg b \wedge \neg a)) \\ &\equiv (a \wedge (\neg b \vee c) \wedge (\neg c \vee b)) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \end{aligned}$$

Applying the identity we just discovered, namely, $X \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y) \equiv (X \wedge Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z)$, for $X \equiv a$, $Y \equiv b$, and $Z \equiv c$.

$$Q(x) \equiv (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$$

but this is exactly $P(x)$. Therefore, the right-hand side may be expressed $\{x : P(x)\}$, which is precisely the left-hand side. □

Chapter 6

The Greatest Common Divisor

6.1 Warm-Up Exercises

Warm-Up Exercise 6.1. What is the remainder when -98 is divided by 7?

Solution. $-98\ 7 = -14$, so the remainder is 0. □

Warm-Up Exercise 6.2. Calculate $\gcd(10, -65)$.

Solution. We have $10 = 2 \cdot 5$ and $-65 = -1 \cdot 5 \cdot 13$, so the GCD is 5. □

Warm-Up Exercise 6.3. Let $a, b, c \in \mathbb{Z}$. Consider the implication S: If $\gcd(a, b) = 1$ and $c \mid (a + b)$, then $\gcd(b, c) = 1$. Fill in the blanks to complete a proof of S.

(a) Since $\gcd(a, b) = 1$, by Bézout's Lemma, there exist integers x and y such that $ax + by = 1$.
(b) Since $c \mid (a + b)$, by definition, there exists an integer k such that $a + b = ck$.
(c) Substituting $a = ck - b$ into the first equation, we get $1 = (ck - b)x + by = b(-x + y) + c(kx)$.
(d) Since 1 is a common divisor of b and c and $-x + y$ and kx are integers, $\gcd(b, c) = 1$ by the GCD Characterization Theorem.

Warm-Up Exercise 6.4. Disprove: For all integers a, b, and c, if $a \mid (bc)$, then $a \mid b$ or $a \mid c$.

Proof. We prove the negation, there are integers a, b, and c where $a \mid bc$, $a \nmid b$, and $a \nmid c$.

Let $a = 15$, $b = 5$, and $c = 3$. Clearly, $a \nmid b$ and $a \nmid c$. However, $bc = 15$, and $15 \mid 15$. □

6.2 Recommended Problems

Recommended Problem 6.1.

(a) Use the Extended Euclidean Algorithm to find three integers x, y and $d = \gcd(1112, 768)$ such that $1112x + 768y = d$.

Solution. Apply the EEA with $x = 1112$ and $y = 768$.

x	y	r	q
1	0	1112	
0	1	768	
1	-1	344	1
-2	3	80	2
9	-13	24	4
-29	42	8	3
96	-139	0	3

Therefore, we have that $d = \gcd(1112, 768) = 8$, and that

$$1112(-29) + 768(42) = 8$$

That is, our solution is when $x = -29$ and $y = 42$. □

(b) Determine integers s and t such that $768s - 1112t = \gcd(768, -1112)$.

Solution. Since the GCD is invariant under sign changes, we immediately know that $\gcd(768, -1112) = 8$. We also have that $1112(-29) + 768(42) = 8$. But this is the same as saying $768(42) - 1112(29) = 8$, so $s = 42$ and $t = 29$. □

Recommended Problem 6.2. Prove that for all $a \in \mathbb{Z}$, $\gcd(9a + 4, 2a + 1) = 1$.

Proof. Let a be an integer. We must show that $9a+4$ and $2a+1$ are coprime. Recall the Coprimeness Characterization Theorem: it suffices to find integers a and b such that $(9a + 4)a + (2a + 1)b = 1$.

Choose $a = -2$ and $b = 9$. Then,

$$\begin{aligned}(9a + 4)a + (2a + 1)b &= -2(9a + 4)a + 9(2a + 1)\\ &= -18a - 8 + 18a + 9\\ &= 1\end{aligned}$$

as desired. Therefore, $\gcd(9a + 4, 2a + 1) = 1$. □

Recommended Problem 6.3. Let $\gcd(x, y) = d$ for integers x and y. Express $\gcd(18x + 3y, 3x)$ in terms of d and prove that you are correct.

Proof. Let x and y be integers with GCD d.

We may apply GCD With Remainders to reduce $g = \gcd(18x + 3y, 3x)$. We have $18x + 3y = 6(3x) + 3y$, so $g = \gcd(3x, 3y)$.

Now, $d \mid x$ and $d \mid y$, so we can find integers m and n where $x = dm$ and $y = dn$. Multiplying through by 3, we have $3x = (3d)m$ and $3y = (3d)n$. It follows that $3d \mid 3x$ and $3d \mid 3y$, that is, $3d$ is a common divisor of $3x$ and $3y$.

By Bézout's Lemma, there are integers s and t where $xs + yt = d$. Again multiplying through by 3, we have $(3x)s + (3y)t = 3d$.

Therefore, by the GCD Characterization Theorem, $\gcd(3x, 3y) = 3d$. □

Recommended Problem 6.4. Let $a, b \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$, then $\gcd(2a + b, a + 2b) \in \{1, 3\}$.

Proof. Let a and b be coprime integers.

Applying GCD WR, we have that $2a + b = 2(a + 2b) - 3b$, so $\gcd(2a + b, a + 2b) = \gcd(a + 2b, -3b)$. The properties of GCD state this is equivalent to $\gcd(3b, a + 2b)$.

The GCD of $3b$ and $a + 2b$ must divide both $3b$ and $a + 2b$. The positive divisors of $3b$ are 1, 3, and any positive divisor $d \geq 2$ of b. We show that no such divisors of b also divide $a + 2b$.

Suppose for a contradiction that an integer $d \geq 2$ divides both b and $a + 2b$. Then, by DIC, $d \mid ((a + 2b) - 2(b))$, that is, $d \mid a$. This means that d is a common divisor of a and b. However, a and b are coprime, meaning $d = 1$. This is a contradiction since $1 \not\geq 2$. Therefore, no positive divisor of b, other than 1, also divides $a + 2b$.

It follows that $\gcd(2a + b, a + 2b)$ can only be 1 or 3, as desired. □

Recommended Problem 6.5. Prove that for all integers a, b and k, if $b \neq 0$, then $\gcd(a, b) \leq \gcd(ak, b)$.

Proof. Let a, b, and k be integers where b is non-zero. Also, let $d = \gcd(a, b)$ and $g = \gcd(ak, b)$. We must show $d \leq g$.

We will apply the GCD from Prime Factorization. For convenience, we define p_n to be the n^{th} prime. First, by UPF, we are guaranteed to be able to write $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, and $k = p_1^{\kappa_1} p_2^{\kappa_2} \cdots p_n^{\kappa_n}$, with non-negative α_i, β_i, and κ_i. Notice that we may write ak as a product of primes: $p_1^{\alpha_1 + \kappa_1} p_2^{\alpha_2 + \kappa_2} \cdots p_n^{\alpha_n + \kappa_n}$.

Now, by GCD PF, we have $d = p_1^{\delta_1} p_2^{\delta_2} \cdots p_n^{\delta_n}$, where $\delta_i = \min(\{\alpha_i, \beta_i\})$ for all integers $1 \leq i \leq k$. Likewise, we have $g = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_n^{\gamma_n}$, where $\gamma_i = \min(\{\alpha_i + \kappa_i, \beta_i\})$.

We will show that $\delta_i \leq \gamma_i$ for all i, from which it follows $d \leq g$. Let i be arbitrary.

If $\alpha_i \leq \beta_i \leq \alpha_i + \kappa_i$, then we have $\delta_i = \alpha_i$ and $\gamma_i = \beta_i$. It follows that $\delta_i \leq \gamma_i$. Otherwise, $\beta_i \leq \alpha_i \leq \alpha_i + \kappa_i$, so $\delta_i = \beta_i$ and $\kappa_i = \alpha_i$. We again have $\delta_i \leq \gamma_i$.

Therefore, since every exponent in the prime factorization of d is less than or equal to the corresponding exponent in the prime factorization of g, it must be the case that $d \leq g$. □

Recommended Problem 6.6. Prove that for all integers a, b and c: if $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$.

Proof. Let a, b, and c be integers such that a and b divide c, and a and b are coprime.

Then, there exist integers m and n such that $am = c$ and $bn = c$. Also, by the CCT, there exist integers s and t such that $as + bt = 1$.

Then, $cas + cbt = c$, so $(bn)as + (am)bt = c$. It follows that $ab(ns + mt) = c$, so $ab \mid c$. □

Recommended Problem 6.7. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid a$, then $\gcd(b, c) = 1$.

Proof. Let a, b, and c be integers such that $\gcd(a, b) = 1$ and $c \mid a$.

Then, $nc = a$ for some integer n and, by Bézout's Lemma, $as + bt = 1$. Substituting, $(nc)a + bt = bt + c(na) = 1$ for integers t and na, so by the CCT, $\gcd(b, c) = 1$. □

Recommended Problem 6.8. Let a and b be integers. Prove that if $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for all $m, n \in \mathbb{N}$. You may use the result which is proved in Example 14 in the notes.

Proof. Recall that Example 14 proved that for all integers a, b, and natural numbers n, if $\gcd(a, b) = 1$, then $\gcd(a, b^n) = 1$. Therefore, it suffices to let $c = b^n$ and prove that $\gcd(a, c) = 1$ implies $\gcd(a^m, c) = 1$.

In fact, we may simplify the problem further. If we show that the arguments of the GCD are commutative, then we may again use the result from Example 14. Let x and y be coprime integers, that is, $\gcd(x, y) = 1$. By Bézout's Lemma, there exist s and t such that $xs + yt = 1$. Equivalently, $yt + xs = 1$, and by the CCT, $\gcd(y, x) = 1$.

Then, $\gcd(a, c) = \gcd(c, a) = 1$. By Example 14, $\gcd(c, a^m) = 1$, that is, $\gcd(a^m, c) = \gcd(a^m, b^n) = 1$, as desired. □

Recommended Problem 6.9. Suppose a, b and n are integers. Prove that $n \mid \gcd(a, n) \cdot \gcd(b, n)$ if and only if $n \mid ab$.

Proof (sooshi, CS Discord). Let a, b, and n be integers. Then, let $d = \gcd(a, n)$ and $c = \gcd(b, n)$. We prove both implications.

($\Rightarrow$) Suppose that $n \mid dc$. Recall that by definition, $d \mid a$ and $c \mid b$. Then, we may write $dn = a$ and $cm = b$ for some integers n and m. Multiplying together, $dc(mn) = ab$, that is, since mn is an integer, $dc \mid ab$. By the transitivity of divisibility, $n \mid dc$ and $dc \mid ab$ imply $n \mid ab$, as desired.

($\Leftarrow$) Suppose that $n \mid ab$. We apply Bézout's Lemma to rewrite $d = as + nt$ and $c = bx + ny$ with integers s, t, x, and y. Multiplying together gives $dc = absx + asny + bxnt + n^2ty$. This factors to $dc = (ab)(sx) + n(asy + bxt + nty)$. Since we have both $n \mid ab$ and $n \mid n$, by DIC, $n \mid (ab)(sx) + n(asy + bxt + nty)$. However, this is just $n \mid dc$.

Therefore, since both implications hold, $n \mid dc$ if and only if $n \mid ab$. □

Recommended Problem 6.10. How many positive divisors does 33480 have?

Solution. We may apply prime factorization to get $33480 = 2^3 \cdot 3^3 \cdot 5 \cdot 31$. Then, by DFPF, we have that any positive divisor $d = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 31^\delta$ for integers $0 \leq \alpha \leq 3$, $0 \leq \beta \leq 3$, $0 \leq \gamma \leq 1$, and $0 \leq \delta \leq 1$.

That is, there are 4 choices for each of α and β, and 2 choices for γ and δ. Multiplying out, we have $4 \cdot 4 \cdot 2 \cdot 2 = 64$ positive divisors. □

Recommended Problem 6.11. Prove that for all integers a and b, if $9a^2 = b^4$ where $a, b \in \mathbb{Z}$, then 3 is a common divisor of a and b.

Proof. Let a and b be integers such that $9a^2 = b^4$. Without loss of generality, let both a and b be positive (if $a = b = 0$, then, trivially, $3 \mid a$ and $3 \mid b$).

By UFT, $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for k distinct primes p_i and non-negative integers α_i. Likewise, $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ for non-negative integers β_i. Since 3 is prime, there is an n where $p_n = 3$.

It follows that $9a^2$ has $2 + 2\alpha_n$ factors of 3 and that b^4 has $4\beta_n$ factors. Since $9a^2 = b^4$, by UFT, $2 + 2\alpha_n = 4\beta_n$.

We have that $4\beta_n = 2 + 2\alpha_n \geq 2$, so $\beta_n \geq 1$, which means $3 \mid b$.

However, if $\beta_n \geq 1$, then $2 + 2\alpha_n = 4\beta_n \geq 4$, which means $\alpha_n \geq 1$. That is, $3 \mid a$.

Therefore, 3 is a common divisor of a and b. □

Recommended Problem 6.12. Let $n \in \mathbb{N}$. Prove that if p is prime and $p \leq n$, then p does not divide $n! + 1$.

Proof. Let n be a natural number, and p be a prime number.

Since $n!$ is defined as the product of all positive integers up to n and $p \leq n$, p clearly divides n. Therefore, $n! = kp$ for some integer k. Then, k is the product of all positive integers up to n *except* p. Since p is prime, $k \nmid p$.

Then, we have $n! + 1 = p(k + \frac{1}{p})$, so $p \mid (n! + 1)$ only if $k + \frac{1}{p}$ is an integer, which it clearly is not (since $p \geq 2$). Therefore, $p \nmid (n! + 1)$. □

6.3 Challenges

Challenge 6.1. Prove that for any integer $a \neq 1$ and $n \in \mathbb{N}$, $\gcd\left(\frac{a^n-1}{a-1}, a-1\right) = \gcd(n, a-1)$.

Challenge 6.2. Let n be a positive integer for which $\gcd(n, n+1) < \gcd(n, n+2) < \cdots < \gcd(n, n+20)$. Prove that $\gcd(n, n+20) < \gcd(n, n+21)$.

Challenge 6.3. Let a and b be nonnegative integers. Prove that $\gcd(2^a - 1, 2^b - 1) = 2\gcd(a, b) - 1$.

Challenge 6.4. An integer n is *perfect* if the sum of all of its positive divisors (including 1 and itself) is $2n$.

(a) Is 6 a perfect number? Give reasons for your answer.

(b) Is 7 a perfect number? Give reasons for your answer.

(c) Prove the following statement: If k is a positive integer and $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is perfect.

Challenge 6.5. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a^n, b^n) = \gcd(a, b)^n$ for all $n \in \mathbb{N}$.

Chapter 7

Linear Diophantine Equations

7.1 Warm-Up Exercises

Warm-Up Exercise 7.1. Find the complete integer solution to $7x + 11y = 3$.

Solution. Begin by applying the EEA to determine one solution for x and y:

x	y	r	q
1	0	7	0
0	1	11	0
-1	1	4	-1
2	-1	3	2

which gives $7(2) + 11(-1) = 3$. Since 7 and 11 are prime, we immediately know their GCD is 1. Now, apply the LDET to determine the complete solution set:

$$\{(x, y) : x = 2 + 11n, y = -1 - 7n, n \in \mathbb{Z}\}$$ □

Warm-Up Exercise 7.2. Find the complete integer solution to $28x + 60y = 10$.

Solution. Begin by applying the EEA to find the GCD:

y	x	r	q
1	0	60	0
0	1	28	0
1	-4	4	2
-7	29	0	7

Therefore, $\gcd(28, 60) = 4$. However, $4 \nmid 10$, so there are no solutions to this equation. □

7.2 Recommended Problems

Recommended Problem 7.1. Find all non-negative integer solutions to $12x + 57y = 423$.

Solution. Since $12 = 3 \times 4$ and $57 = 3 \times 19$, clearly $\gcd(12, 57) = 3$. We also have that $423 \mid 3$, so solutions exist. Applying EEA, we have

y	x	r	q
1	0	57	0
0	1	12	0
1	-4	9	4
-1	5	3	1

so our base solution is $12(5) + 57(-1) = 3$. Multiplying through by $\frac{423}{3} = 141$, we have $12(705) + 57(-141) = 423$. By the LDET, we arrive at our solution set in the integers:

$$\{(x, y) : x = 705 + 19n, y = -141 - 4n, n \in \mathbb{Z}\}$$

However, we want to restrict $x \geq 0$ and $y \geq 0$. Notice that $x \geq 0$ when $n \geq -\frac{705}{19}$, that is, $n \geq -37$. Likewise, $y \geq 0$ when $n \leq -\frac{141}{35}$, that is, $n \leq -36$.

This just means that $-37 \leq n \leq -36$, or $n = -37, -36$. Therefore, the solution set is $(x, y) \in \{(2, 7), (21, 3)\}$. □

Recommended Problem 7.2. Prove or disprove the following implications:

(a) For all integers a, b, and c, if there exists an integer solution to $ax^2+by^2 = c$, then $\gcd(a, b) \mid c$.

Proof. Let a, b, and c be integers. Suppose there is an integer solution in x and y to the equation $ax^2 + by^2 = c$. Since x^2 and y^2 are integers, this is a solution to the equation $as + bt = c$ with integers $s = x^2$ and $t = y^2$.

It immediately follows from the LDET that $\gcd(a, b) \mid c$. □

(b) For all integers a, b, and c, if $\gcd(a, b) \mid c$, then there exists an integer solution to $ax^2+by^2 = c$.

Proof. Consider the counterexample where $a = b = 1$ and $c = -2$. We have that $\gcd(a, b) = \gcd(1, 1) = 1$ and clearly $1 \mid -2$.

We now have the equation $(1)x^2 + (1)y^2 = -2$. From the properties of integers, $x^2 \geq 0$ and $y^2 \geq 0$, so $x^2 + y^2 \geq 0$. Then, $x^2 + y^2 \geq 0$ but -2 is not non-negative. Therefore, no solutions to $x^2 + y^2 = -2$ exist. □

Recommended Problem 7.3. Consider the following statement: For all integers a, b, c, and x_0, there exists an integer y_0 such that $ax_0 + by_0 = c$.

(a) Carefully write down the negation of this statement and prove that this negation is true.

Proof. We prove the negation:

There exist integers a, b, c, and x_0 such that for all integers y_0, $ax_0 + by_0 \neq c$.

Select $a = x_0 = 1$, $b = 0$, and $c = 2$. Let y_0 be an integer. We must show that $(1)(1)+(0)y_0 \neq (2)$. This is just $1 \neq 2$, which is true independent of y_0. □

(b) Let $a, b, c \in \mathbb{Z}$. Fill in the blank to make the following statement true and prove that it is true. $\boxed{b \text{ is non-zero, } b \mid a, \text{ and } b \mid c}$ if and only if for all integers x_0, there exists an integer y_0 such that $ax_0 + by_0 = c$.

Proof. Let a, b, and c be integers.

We prove the biconditional by proving both implications.

($\Rightarrow$) Suppose b is non-zero, $b \mid a$, and $b \mid c$. We break into cases on a:

If $a = 0$, then we must show that there exists a y_0 such that $by_0 = c$. This follows immediately from the fact that $b \mid c$.

If a is non-zero, it follows that $\gcd(a, b) = |b|$. Then, since $b \mid c$, we have $\gcd(a, b) \mid c$. We may now apply the LDET. The solution set to the linear Diophantine equation $ax_0 + by_0 = c$ is

$$\{(x_0, y_0) : x_0 = x + \frac{b}{|b|}n, y_0 = y + \frac{a}{|b|}n, n \in \mathbb{Z}\}$$

for some initial solution (x, y). Since n ranges through all integers, we may drop the absolute value bars. Then, $x_0 = x + n$, so every integer x_0 appears in the solution set at $n = x_0 - x$, with a corresponding y_0.

Alternatively stated, for every integer x_0, there exists a y_0 such that $ax_0 + by_0 = c$.

($\Leftarrow$) Suppose that for all integers x_0, we may choose an integer y_0 so $ax_0 + by_0 = c$. Let x_0 be an integer.

Suppose for a contradiction that $b = 0$, so $ax_0 = c$. This is clearly not true for all a, c, and x_0. Therefore, b is non-zero.

Now, break into cases on a. Suppose that $a = 0$. Then, we may find y_0 such that $by_0 = c$, which is the same as saying $b \mid c$.

Suppose that a is non-zero. Since both a and b are non-zero and $ax_0 + by_0 = c$ is a solution to the LDE $ax + by = c$, the LDET applies, giving $\gcd(a, b) \mid c$.

However, since the LDET applies, there is an entire solution set given by

$$\{(x, y) : x = x_0 + \frac{b}{\gcd(a, b)}n, y = y_0 + \frac{a}{\gcd(a, b)}n, n \in \mathbb{Z}\}$$

Now, recall that x_0 is an arbitrary integer. Therefore, the values of x given in the set above must also span the integers, that is, any arbitrary integer x may be written $x_0 + \frac{b}{\gcd(a,b)}n$.

This implies that $\frac{b}{\gcd(a,b)} = 1$, that is, $b = \gcd(a, b)$, since GCD is positive.

Therefore, b is non-zero, $\gcd(a, b) = b$ divides c, and by definition, b divides a. □

Recommended Problem 7.4. Suppose a and b are integers. Prove that $\{ax + by : x, y \in \mathbb{Z}\} = \{n \gcd(a, b) : n \in \mathbb{Z}\}$.

Proof. Let a and b be integers with GCD d. We prove $\{ax + by : x, y \in \mathbb{Z}\} = \{nd : n \in \mathbb{Z}\}$ by mutual containment.

($\subseteq$) Let x and y be integers. Then, since $d \mid a$ and $d \mid b$, $d \mid (ax + by)$. This means we may write $ax + by$ as nd, as desired.

($\supseteq$) Let n be an integer. By Bézout's Lemma, we may write $d = xs + yt$ for integers s and t. Multiplying through by n, we have $nd = (ns)x + (nt)y$. We may let $a = ns$ and $b = nt$, which are integers, and have $nd = ax + by$ as desired.

Therefore, since the sets are mutually contained, they are equal. □

Note: This is essentially a restatement of Jerry Wang's GCD derivation by subgroups.

7.3 Challenge

Challenge 7.1. For how many integer values of c does $8x + 5y = c$ have exactly one solution where both x and y are strictly positive integers?

Chapter 8

Congruence and Modular Arithmetic

8.1 Warm-Up Exercises

Warm-Up Exercise 8.1. Is 738645899999992324343123 divisible by 11?

Solution. We may simply apply Proposition 9 from the course notes: an integer is divisible by 11 if the difference of the sums of the even and odd digits is divisible by 11.

The even digits are $7+8+4+8+9+9+9+9+3+4+4+1+3 = 78$ and the odd digits are $3+6+5+9+9+9+9+2+2+3+3+2 = 62$. We have $78-62 = 16$ which is not divisible by 11.

Therefore, $11 \nmid 738645899999992324343123$. □

Warm-Up Exercise 8.2. For each linear congruence, determine the complete solution, if a solution exists.

(a) $3x \equiv 11 \pmod{18}$

Solution. Notice that $\gcd(3, 18) = 3$ and $3 \nmid 11$. Therefore, by LCT, there are no solutions. □

(b) $4x \equiv 5 \pmod{21}$

Solution. Notice that $\gcd(4, 21) = 1$ and $1 \mid 5$. Therefore, LCT guarantees a set of solutions where $x \equiv x_0 \pmod{21}$ for some particular solution x_0.

By inspection, $21 + 4(-4) = 5$, so $4(-4) \equiv 5 \pmod{21}$.

Therefore, the set of solutions is $x \in [-4]_{21} = [17]_{21}$. □

Warm-Up Exercise 8.3. Complete the addition and multiplication tables for $\mathbb{Z}_5$.

Solution. The elements of $\mathbb{Z}_5$ are $\{[0],[1],[2],[3],[4]\}$:

$+$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[1]$	$[1]$	$[2]$	$[3]$	$[4]$	$[0]$
$[2]$	$[2]$	$[3]$	$[4]$	$[0]$	$[1]$
$[3]$	$[3]$	$[4]$	$[0]$	$[1]$	$[2]$
$[4]$	$[4]$	$[0]$	$[1]$	$[2]$	$[3]$

$\times$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[2]$	$[0]$	$[2]$	$[4]$	$[1]$	$[3]$
$[3]$	$[0]$	$[3]$	$[1]$	$[4]$	$[2]$
$[4]$	$[0]$	$[4]$	$[3]$	$[2]$	$[1]$

□

Warm-Up Exercise 8.4. What is the remainder when 14^{43} is divided by 41?

Solution. Since 41 is prime and $41 \nmid 14$, we may apply Fermat's Little Theorem.

$$14^{41-1} = 14^{40} \equiv 1 \pmod{41}$$

Now, simply apply modular arithmetic: $14^2 = 196 \equiv -9 \pmod{41}$, and $14^3 = 14 \cdot 14^2 \equiv 14 \cdot -9 \equiv -3 \pmod{41}$. Finally, $14^{40} \cdot 14^3 = 14^{43} \equiv 1 \cdot -3 \equiv 38 \pmod{41}$. Therefore, the remainder is 38. □

Warm-Up Exercise 8.5. Solve

$$x \equiv 7 \pmod{11}$$
$$x \equiv 5 \pmod{12}$$

Solution. We apply the Chinese Remainder Theorem since $\gcd(11,12) = 1$. Solutions to the first equation are $x \equiv 7, 18, 29, 40, 51, 62, 73, 84, 95, 106, 117, 128 \pmod{132}$. Solutions to the second are $x \equiv 5, 17, 29, 41, 53, 65, 77, 89, 101, 113, 125 \pmod{132}$. The unique solution common to these is $x \equiv 29 \pmod{132}$. □

Warm-Up Exercise 8.6. What is the smallest non-negative integer x such that $2000 \equiv x \pmod{37}$?

Solution. Simply reduce using the division algorithm, which guarantees a minimal non-negative remainder below 37: we have $2000 = 37(54) + 2$, so $2000 \equiv 2 \pmod{37}$. □

8.2 Recommended Problems

Recommended Problem 8.1. Is $27^{129} + 61^{40}$ divisible by 14? Justify your answer.

Solution. We simplify with $27 \equiv -1 \pmod{14}$ and $61 \equiv 5 \pmod{14}$:

$$\begin{aligned} 27^{129} + 61^{40} &\equiv (-1)^{128+1} + (5)^{40} \pmod{14} \\ &\equiv -1 + 5^{32+8} \pmod{14} \end{aligned}$$

Now, we can repeatedly square 5 to calculate 5^{32} and 5^8.

$$\begin{aligned} 5^2 &\equiv 25 \equiv 11 \pmod{14} \\ 5^4 &\equiv 121 \equiv 9 \pmod{14} \\ 5^8 &\equiv 81 \equiv 11 \pmod{14} \\ 5^{16} &\equiv 121 \equiv 9 \pmod{14} \\ 5^{32} &\equiv 81 \equiv 11 \pmod{14} \end{aligned}$$

Substituting back,

$$27^{129} + 61^{40} \equiv -1 + (11)(11) \equiv 120 \equiv 8 \pmod{14}$$

Therefore, the remainder is 8 by CTR, which is not 0, so $14 \nmid (27^{129} + 61^{40})$. □

Recommended Problem 8.2. Prove Congruence Power (CP): For all positive integers n and integers a and b, if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$.

Proof. Let a and b be integers congruent mod m. We prove by induction on n. Let $P(n)$ denote the statement that $a^n \equiv b^n \pmod{m}$.

The base case $P(1)$, $a^1 \equiv b^1 \pmod{m}$ follows from the hypothesis.

Now, let k be a positive integer. Suppose $P(k-1)$ holds, that is, $a^{k-1} \equiv b^{k-1} \pmod{m}$. Since $a \equiv b \pmod{m}$, we may write $a = b + pm$ for some integer p. By our inductive hypothesis, we write $a^{k-1} = b^{k-1} + qm$ for some integer q.

Multiplying these equations together,

$$\begin{aligned} (a)(a^{k-1}) &= (b + pm)(b^{k-1} + qm) \\ a^k &= b^k + b^{k-1}pm + bqm + pqm^2 \\ a^k - b^k &= (b^{k-1}p + bq + pqm)m \end{aligned}$$

which, since $b^{k-1}p + bq + pqm$ is an integer, implies m divides $a^k - b^k$. By the definition of congruence, $a^k \equiv b^k \pmod{m}$, which is exactly $P(k)$.

Therefore, by induction, $P(n)$ is true for all positive integer n. □

Recommended Problem 8.3. What is the remainder when 3141^{2001} is divided by 17?

Solution. First, notice that $3141 \equiv 13 \pmod{17}$. We also have that $3141^2 \equiv 13^2 \equiv -1 \pmod{17}$. Therefore, $3141^{2001} \equiv 3141(3141^2)^{1000} \equiv 13(-1)^{1000} \equiv 13 \pmod{17}$. By CTR, the remainder is 13. □

Recommended Problem 8.4. Solve $49x^{177} + 37x^{26} + 3x^2 + x + 1 \equiv 0 \pmod{7}$.

Solution. First, notice that $49x^{177} \equiv 0 \pmod{7}$ for any integer x since $7 \mid 49$. Also, $37 \equiv 2 \pmod{7}$, so $37x^{26} \equiv 2x^{26} \equiv 2(x^2)^{13} \pmod{7}$ for any integer x. Additionally, by CFlT, $x^7 \equiv x \pmod{7}$ for any integer x, so $2x^{26} \equiv 2x^{7(3)+5} \equiv 2(x^7)^3(x^5) \equiv 2x^8 \equiv 2x^2 \pmod{7}$ for any integer x.

Now, simply test every value of x and find $5x^2 + x$:

$x \pmod 7$	0	1	2	3	4	5	6
$x^2 \pmod 7$	0	1	4	2	2	4	1
$5x^2 \pmod 7$	0	5	6	3	3	6	5
$5x^2 + x \pmod 7$	0	6	1	6	0	4	4

Now, since $-1 \equiv 6 \pmod{7}$, our solutions are $x \equiv 1, 3 \pmod{7}$. □

Recommended Problem 8.5. Solve

$$\begin{aligned} 3x - 2 &\equiv 7 \pmod{11} \\ 5 &\equiv 4x - 1 \pmod{9} \end{aligned}$$

Solution. We can simplify these congruences by CAM to $3x \equiv 9 \pmod{11}$ and $4x \equiv 6 \pmod{9}$. Since 11 is prime, we can apply CD to the first congruence to get $x \equiv 3 \pmod{11}$. Then, $x = 11k+3$ for some integer k. Substituting,

$$\begin{aligned} 4(11k + 3) &\equiv 6 \pmod{9} \\ 44k &\equiv -6 \pmod{9} \\ -k &\equiv -6 \pmod{9} \\ k &\equiv 6 \pmod{9} \end{aligned}$$

Therefore, $k = 9n + 6$ for an integer n, and $x = 11(9n + 6) + 3 = 99n + 69$. Equivalently, by definition, $x \equiv 69 \pmod{99}$. □

Recommended Problem 8.6. The Chinese Remainder Theorem deals with the case where the moduli are coprime. We now investigate what happens if the moduli are not coprime.

(a) Consider the following two systems of linear congruences:

$$A: \begin{cases} n \equiv 2 & \pmod{12} \\ n \equiv 10 & \pmod{18} \end{cases} \qquad B: \begin{cases} n \equiv 5 & \pmod{12} \\ n \equiv 11 & \pmod{18} \end{cases}$$

Determine which one has solutions and which one has no solutions. For the one with solutions, give the complete solutions to the system. For the one with no solutions, explain why no solutions exist.

Solution. Consider system A. By definition, numbers n congruent to 10 modulo 18 are of the from $n = 10 + 18k$ for some integer k. Substituting into the first congruence, $10 + 18k \equiv 2 \pmod{12}$, that is, $6k \equiv 4 \pmod{12}$. However, since $\gcd(6, 12) = 6$ and $6 \nmid 4$, there are no valid values of k.

Consider system B. By definition, solutions to the second congruence are of the from $n = 11 + 18k$ for some integer k. Substituting, $11 + 18k \equiv 5 \pmod{12}$, that is, $6k \equiv 6 \pmod{12}$. Since $\gcd(6, 12) = 6$ and $6 \mid 6$, a solution exists. By inspection, $k = 1$ is a solution. By LCT, the set of all solutions is given by $\{k \in \mathbb{Z} : k \equiv 1 \pmod{2}\}$. Therefore, values for k are of the form $2m + 1$ for some integer m. Backsubstituting, $n = 11 + 18(2m + 1) = 29 + 36m$. Equivalently, the solution set for all n is given by

$$\{n \in \mathbb{Z} : n \equiv 29 \pmod{36}\}$$ □

(b) Let a_1 and a_2 be integers, and let m_1 and m_2 be positive integers. Consider the following system of linear congruences:

$$S: \begin{cases} n \equiv a_1 & \pmod{m_1} \\ n \equiv a_2 & \pmod{m_2} \end{cases}$$

Using your observations in (a), complete the following two statements. The system S has a solution if and only if $\boxed{a_1 \equiv a_2 \pmod{\gcd(m_1, m_2)}}$. If n_0 is a solution to S, then the complete solution is $\boxed{n \equiv n_0 \pmod{\operatorname{lcm}(m_1, m_2)}}$.

(c) Prove the first statement.

Proof. Let a_1 and a_2 be integers and let m_1 and m_2 be positive integers with GCD d. We prove the biconditional by mutual implication.

($\Rightarrow$) Suppose that $a_1 \equiv a_2 \pmod{d}$. Solutions to the first congruence are of the form $n = a_1 + m_1 x$ for some integer x. However, we may write $a_1 = a_2 + dk$ with integer k, so we have $n = a_2 + dk + m_1 x$. Substituting, $a_2 + dk + m_1 x \equiv a_2 \pmod{m_2}$, that is, $m_1 x \equiv -dk \pmod{m_2}$.

By LCT, this has a solution x_0 because $\gcd(m_1, m_2) = d$ and $d \mid -dk$, and all solutions are given by $x = x_0 + m_2 y$ for some integer y. Backsubstituting, $n = a_1 + m_1(x_0 + m_2 y) = a_1 + m_1 x_0 + m_1 m_2 y$. Therefore, the system S has solutions.

($\Leftarrow$) Suppose that S has a solution. Then, there is some n such that $n = a_1 + m_1 p = a_2 + m_2 q$ for integers p and q. Rearranging, $a_1 - a_2 = m_1 p + m_2 q$. This is an LDE in $a_1 - a_2$. By the LDET, it has solutions if and only if $\gcd(m_1, m_2) \mid (a_1 - a_2)$. This is equivalent by definition to saying $a_1 \equiv a_2 \pmod{d}$. □

Recommended Problem 8.7. Solve $x^3 \equiv 17 \pmod{99}$.

Solution. We can split the modulus to simplify the problem: $99 = 3 \times 3 \times 11$. By SMT, we can solve three simultaneous congruences. However, since 3 appears twice, those congruences are redundant. Then, we may equivalently solve the simultaneous congruences

$$x^3 \equiv 17 \equiv 2 \pmod{3} \quad \text{and} \quad x^3 \equiv 17 \equiv 6 \pmod{11}$$

For the first congruence, we make a table

$x \pmod 3$	0	1	2
$x^2 \pmod 3$	0	1	1
$x^3 \pmod 3$	0	1	2

and see from the last row that the solution is all x such that $x \equiv 2 \pmod{3}$. Repeating for the second congruence, we make a table

$x \pmod{11}$	0	1	2	3	4	5	6	7	8	9	10
$x^2 \pmod{11}$	0	1	4	9	5	3	3	5	9	4	1
$x^3 \pmod{11}$	0	1	8	5	9	4	7	2	6	3	10

and see again in the last row that the solution is all x such that $x \equiv 8 \pmod{11}$. Now, we apply the Chinese Remainder Theorem. There must exist some x_0 so the solution set is all x congruent to x_0 modulo 33.

Note that solutions between 0 and 32 that are congruent to 8 modulo 11 are 8, 19, and 30. Of these, only 8 is congruent to 2 modulo 3. Therefore, the solution is all x such that $x \equiv 8 \pmod{33}$, or, equivalently,

$$x \equiv 8, 41, 74 \pmod{99}$$ □

Recommended Problem 8.8. Solve $x^2 + 25x \equiv 54 \pmod{63}$.

Solution. First, notice that $x^2 + 25x - 54$ factors as $(x-2)(x+27)$. Split the modulus as $63 = 7 \times 9$, so by SMT, we can solve two simultaneous congruences:

$$(x-2)(x+27) \equiv 0 \pmod{7} \qquad (x-2)(x+27) \equiv 0 \pmod{9}$$

Since $-27 \equiv 1 \pmod{7}$ and $-27 \equiv 0 \pmod{9}$, we can equivalently write $x \equiv 1, 2 \pmod{7}$ and $x \equiv 0, 2 \pmod{9}$.

Now, since 7 and 9 are coprime, we take all combinations of the above and apply CRT to each pair, obtaining the set of solutions:

- If $x \equiv 1 \pmod{7}$ and $x \equiv 0 \pmod{9}$, then $x \equiv 36 \pmod{63}$
- If $x \equiv 1 \pmod{7}$ and $x \equiv 2 \pmod{9}$, then $x \equiv 29 \pmod{63}$

- If $x \equiv 2 \pmod 7$ and $x \equiv 0 \pmod 9$, then $x \equiv 9 \pmod{63}$
- If $x \equiv 2 \pmod 7$ and $x \equiv 2 \pmod 9$, then $x \equiv 2 \pmod{63}$

Therefore, by CRT, $x \equiv 2, 9, 29, 36 \pmod{63}$ are the only solutions. □

Recommended Problem 8.9. Find the smallest positive integer a such that $5n^{13} + 13n^5 + a(9n) \equiv 0 \pmod{65}$ for all integers n.

Solution. Let n be an integer. Since $65 = 5 \times 13$, we split the congruence with SMT:

$$\begin{aligned} 5n^{13} + 13n^5 + a(9n) &\equiv 0 \pmod 5 \\ 3n^5 + a(4n) &\equiv 0 \pmod 5 \end{aligned} \qquad \begin{aligned} 5n^{13} + 13n^5 + a(9n) &\equiv 0 \pmod{13} \\ 5n^{13} + a(9n) &\equiv 0 \pmod{13} \end{aligned}$$

Now, we apply CFℓT to both congruences to obtain

$$\begin{aligned} 3n + a(4n) &\equiv 0 \pmod 5 \\ (3 + 4a)n &\equiv 0 \pmod 5 \end{aligned} \qquad \begin{aligned} 5n + a(9n) &\equiv 0 \pmod{13} \\ (5 + 9a)n &\equiv 0 \pmod{13} \end{aligned}$$

This has a trivial solution when $n \equiv 0 \pmod{65}$, but since n is arbitrary, we must otherwise have that $3 + 4a \equiv 0 \pmod 5$ and $5 + 9a \equiv 0 \pmod{13}$ by CAD. We solve the simultaneous congruence

$$4a \equiv 2 \pmod 5 \qquad\qquad 9a \equiv 8 \pmod{13}$$

noting that since 5 and 13 are prime, solutions exist. In fact, we can bruteforce to find $a \equiv 3 \pmod 5$ and $a \equiv 11 \pmod{13}$. Then, by the CRT, $a \equiv 63 \pmod{65}$ and the smallest positive such integer is $a = 63$. □

Recommended Problem 8.10. Prove that for distinct primes p and q, $(p^{q-1} + q^{p-1}) \equiv 1 \pmod{pq}$.

Proof. Let p and q be distinct primes. Since $q - 1$ and $p - 1$ are positive integers, p^{q-1} is a multiple of p and q^{p-1} of q. By definition, $p^{q-1} \equiv 0 \pmod p$ and $q^{p-1} \equiv 0 \pmod q$. Since $p \nmid q$ and $q \nmid p$, by FℓT, we have $q^{p-1} \equiv 1 \pmod p$ and $p^{q-1} \equiv 1 \pmod q$. Therefore, by CAM, we have the simultaneous congruences

$$\begin{aligned} (p^{q-1} + q^{p-1}) &\equiv 1 \pmod p \\ (p^{q-1} + q^{p-1}) &\equiv 1 \pmod q \end{aligned}$$

As distinct primes, $\gcd(p, q) = 1$. Then, by SMT, $(p^{q-1} + q^{p-1}) \equiv 1 \pmod{pq}$. □

Recommended Problem 8.11. If a and b are integers, $3 \nmid a$, $3 \nmid b$, $5 \nmid a$, and $5 \nmid b$, prove that $a^4 \equiv b^4 \pmod{15}$

Proof. Let x be an integer where $3 \nmid x$ and $5 \nmid x$. We exhaust possibilities for $x \pmod{15}$:

$x \pmod{15}$	1	2	4	7	8	11	13	14
$x^2 \pmod{15}$	1	4	1	4	4	1	4	1
$x^4 \pmod{15}$	1	1	1	1	1	1	1	1

Therefore, for any integer x neither a multiple of 3 nor 5, $x^4 \equiv 1 \pmod{15}$. It follows that for any two integers a and b, with both neither multiples of 3 nor 5, $a^4 \equiv b^4 \pmod{15}$. □

8.3 Challenge

Challenge 8.1. A basket contains a number of eggs and when the eggs are removed 2, 3, 4, 5 and 6 at a time, there are 1, 2, 3, 4 and 5, respectively, left over. When the eggs are removed 7 at a time there are none left over. Assuming none of the eggs broke during the preceding operations, determine the minimum number of eggs that were in the basket.

Solution. Let n be the number of eggs in the basket. Since removing 6 eggs leaves at least 5 eggs, $n \geq 11$. We interpret the constraints as a system of linear congruences:

$$\begin{aligned} n &\equiv 1 \pmod{2} & n &\equiv 2 \pmod{3} \\ n &\equiv 3 \pmod{4} & n &\equiv 4 \pmod{5} \\ n &\equiv 5 \pmod{6} & n &\equiv 0 \pmod{7} \end{aligned}$$

Note that all multiples of 7 are odd, so we may ignore the first condition. Also, we may write $n = 7m$ for some integer m and simplify:

$$\begin{aligned} 7m \equiv m &\equiv 2 \pmod{3} & 7m \equiv 3m &\equiv 3 \pmod{4} \\ 7m \equiv 2m &\equiv 4 \pmod{5} & 7m \equiv m &\equiv 5 \pmod{6} \end{aligned}$$

Since 3 and 4 are coprime, we apply CD to the second congruence, $m \equiv 1 \pmod{4}$. By definition and the first congruence, $m = 2 + 3p$ for an integer p. Apply CAM and CD:

$$\begin{aligned} 2 + 3p &\equiv 1 \pmod{4} & 2(2 + 3p) &\equiv 4 \pmod{5} & 2 + 3p &\equiv 5 \pmod{6} \\ 3p &\equiv 3 \pmod{4} & 6p &\equiv 0 \pmod{5} & 3p &\equiv 3 \pmod{6} \\ p &\equiv 1 \pmod{4} & p &\equiv 0 \pmod{5} & 3p &\equiv 3 \pmod{6} \end{aligned}$$

We find by inspection that $p = 1$ is a solution to the third congruence, so by LCT, it is equivalently $p \equiv 1 \pmod{2}$ (i.e. p is odd). However, this is also implied by the first congruence, leaving us with two congruences. Since 4 and 5 are coprime, we may apply the Chinese Remainder Theorem. By inspection, $p = 5$ is a solution. Therefore, all p are of the form $p \equiv 5 \pmod{20}$ or $p = 5 + 20q$ for an integer q.

It follows that $m = 2 + 3(5 + 20q) = 17 + 60q$ and that $n = 7(17 + 60q) = 119 + 420q$. Since $119 < 420$, the lowest possible value of n is 119 eggs. □

Chapter 9

The RSA Public-Key Encryption Scheme

9.1 Warm-Up Exercises

Warm-Up Exercise 9.1. Given the public RSA encryption key $(e, n) = (5, 35)$, find the corresponding decryption key (d, n).

Solution. We factor n and find that $n = 5 \times 7$. Therefore, $p = 5$ and $q = 7$.

We can now find the decryption key d by solving $ed \equiv 1 \pmod{(p-1)(q-1)}$:

$$5d \equiv 1 \pmod{24}$$

By inspection, $d = 5$ is a solution. Because we have $1 < 5 < (p-1)(q-1)$, this is in fact the decryption key.

Therefore, the decryption key is $(5, 35)$. □

9.2 Recommended Problems

Recommended Problem 9.1. Suppose that in setting up RSA, Alice chooses $p = 47$, $q = 37$, and $e = 25$.

(a) What is Alice's public key?

Solution. We have $n = pq = 1739$, so Alice's pubkey is $(25, 1739)$. □

(b) What is Alice's private key?

Solution. We solve the congruence $ed \equiv 1 \pmod{(p-1)(q-1)}$ or $25d \equiv 1 \pmod{1656}$ which is equivalent to solving the LDE

$$25d + 1656y = 1$$

We do this with the good 'ole EEA:

y	d	r	q
1	0	1656	
0	1	25	
1	-66	6	66
-4	265	1	4

and conclude that $d = 265$ is a solution to our LDE. Since $1 < 265 < 1656$, it is in fact the decryption key. Therefore, Alice's privkey is $(265, 1739)$. □

(c) Suppose Alice wishes to send Bob the message $M = 20$. Bob's public key is $(23, 377)$ and Bob's private key is $(263, 377)$. What is the cipher text corresponding to M?

Solution. We compute the ciphertext C as $C \equiv M^e \pmod{n}$ where $0 \leq C < n$.

Substituting, $C \equiv 20^{23} \pmod{377}$. We perform the computation by hand like the masochistic math majors we are:

$$\begin{aligned} C &\equiv 20 \times 20^2 \times 20^4 \times 20^{16} \pmod{377} \\ &\equiv 20 \times 23 \times 23^2 \times 23^8 \pmod{377} \\ &\equiv 20 \times 23 \times 152 \times 152^4 \pmod{377} \\ &\equiv 20 \times 23 \times 152 \times 107^2 \pmod{377} \\ &\equiv 83 \times 152 \times 139 \pmod{377} \\ &\equiv 175 \times 139 \pmod{377} \\ &\equiv 197 \pmod{377} \end{aligned}$$

and since we have $0 \leq 197 < 377$, this is indeed our cyphertext. □

Recommended Problem 9.2. Set up an RSA scheme using two-digit prime numbers. Select values for the other variables and test encrypting and decrypting messages.

Solution. Let $p = 11$ and $q = 13$, the smallest two-digit prime numbers. Then, $n = pq = 143$. Choose e coprime to $(p-1)(q-1) = 120$ to be $e = 23$. To generate d, we solve $23d \equiv 1 \pmod{120}$, i.e., $23d + 120y = 1$, with the EEA:

y	d	r	q
1	0	120	
0	1	23	
1	-5	5	5
-4	21	3	4
5	-26	2	1
-9	47	1	1

Therefore, $d = 47$, and we have the pubkey $(23, 143)$ and privkey $(47, 143)$.

Suppose we want to send the ASCII exclamation mark "!", $M = 33$. Then, we compute the ciphertext $C \equiv M^e \pmod{n}$, i.e., $C \equiv 33^{23} \pmod{143}$. Expanding and reducing to the remainder, $C = 132$.

We decrypt by taking $R \equiv C^d \pmod{n}$, i.e., $R \equiv 132^{47} \pmod{143}$. Since in decryption we know p and q, we equivalently solve both

$$R \equiv 132^{47} \pmod{11} \quad \text{and} \quad R \equiv 132^{47} \pmod{13}$$

Simplifying by FℓT, we obtain

$$\begin{aligned} R &\equiv 132^7 \equiv 0 \pmod{11} \\ R &\equiv 132^{11} \equiv 7 \pmod{13} \end{aligned}$$

By the CRT, there is a unique solution modulo 143. We notice by inspection that $13(2) + 7 = 33 = 11(3)$, so $R = 33$ is the received message. □

9.3 Challenge

Challenge 9.1. Write a computer program to implement RSA encryption and decryption.

Solution. Allow me to demonstrate just how overpowered Wolfram Mathematica is:

```
(* Generates RSA keypair by default *)
keys = GenerateAsymmetricKeyPair[];
msg = "This is cheating";
cyphertext = Encrypt[keys["PublicKey"], msg];
received = Decrypt[keys["PrivateKey"], cyphertext];
```

Oh, you meant actually do the calculations? Okay.

```
(* Generate random primes below 100 *)
{p, q} = RandomPrime[100, 2]; n = p*q;
(* Generate e as a random coprime *)
m = (p-1)(q-1);
e = RandomChoice@Pick[Range[m], CoprimeQ[m, Range[m]]];
(* Solve d automagically *)
d = D /. Solve[e*D == 1, D, Modulus -> 120][[1]];

(* Sample encryption/decryption of 42 *)
C = PowerMod[42,e,n];
R = PowerMod[C,d,n];
```

□

Chapter 10

Complex Numbers

10.1 Warm-Up Exercises

Warm-Up Exercise 10.1. Express $\dfrac{2-i}{3+4i}$ in standard form.

Solution. Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{2-i}{3+4i} = \frac{(2-i)(3-4i)}{9+16} = \frac{2-11i}{25} = \frac{2}{25} - \frac{11}{25}i$$

□

Warm-Up Exercise 10.2. Write $x = \dfrac{9+i}{5-4i}$ in polar form, $r(\cos\theta + i\sin\theta)$, with $0 \leq \theta < 2\pi$.

Solution. We express first in standard form by multiplying through the conjugate:

$$\frac{9+i}{5-4i} = \frac{(9+i)(5+4i)}{41} = \frac{41+41i}{41} = 1+i$$

We can geometrically interpret this as $\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$. □

Warm-Up Exercise 10.3. Write $(\sqrt{3}+i)^4$ in standard form.

Solution. We first place the quantity within the brackets in polar form. By inspection, this is $2\,\mathrm{cis}\,\frac{\pi}{6}$. Now, applying DMT, we have $(2\,\mathrm{cis}\,\frac{\pi}{6})^4 = 2^4\,\mathrm{cis}^4\,\frac{\pi}{6} = 16\,\mathrm{cis}\,\frac{2\pi}{3}$.

Expressing in standard form, $16(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 16(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -8 + 8\sqrt{3}i$ □

Warm-Up Exercise 10.4. Find all $z \in \mathbb{C}$ such that $z^5 = 1$ and plot the solutions in the complex plane.

(You may state values in polar form.)

Solution. Note that $1 = 1 \operatorname{cis} 0$. Applying the CRNT, we have that the five roots are given by $\sqrt[5]{1} \operatorname{cis}\left(\frac{2k\pi}{n}\right)$ for $k = 0, 1, 2, 3, 4$. These values are $\{1, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{6\pi}{5}, \operatorname{cis}\frac{8\pi}{5}\}$. I am too lazy to learn `tikz` to draw the diagram. □

Warm-Up Exercise 10.5. Find all $z \in \mathbb{C}$ such that $z^2 = \dfrac{1+i}{1-i}$.

Solution. Simplifying the fraction on the right-hand side, $\frac{(1+i)(1+i)}{2} = \frac{1+2i-1}{2} = i$. On the complex plane, $i = 1 \operatorname{cis}\frac{\pi}{2}$. Then, by CRNT, the solutions are $\operatorname{cis}\frac{\pi}{4}$ and $\operatorname{cis}\frac{5\pi}{4}$. Evaluating to get standard form, we have $z = \pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$. □

10.2 Recommended Problems

Recommended Problem 10.1. Express the following complex numbers in standard form.

(a) $\dfrac{(\sqrt{2}-i)^2}{(\sqrt{2}+i)(1-\sqrt{2}i)}$

Solution. Multiply through conjugates of the denominator:

$$\begin{aligned}\frac{(\sqrt{2}-i)^2}{(\sqrt{2}+i)(1-\sqrt{2}i)} &= \frac{(1-2\sqrt{2}i)(\sqrt{2}-i)(1+\sqrt{2}i)}{(3)(3)} \\ &= \frac{(5-\sqrt{2}i)(\sqrt{2}-i)}{9} \\ &= \frac{4\sqrt{2}-7i}{9} \\ &= \frac{4\sqrt{2}}{9} - \frac{7}{9}i\end{aligned}$$

□

(b) $(\sqrt{5}-i\sqrt{3})^4$

Solution. Let $z = \sqrt{5}-i\sqrt{3}$. We have $z^2 = 5 - 2\sqrt{15}i - 3 = 2 - 2\sqrt{15}i$. Finally, $z^4 = (z^2)^2 = 4 - 8\sqrt{15}i - 60 = -56 - 8\sqrt{15}i$. □

Recommended Problem 10.2. Prove all of the Properties of Complex Arithmetic that were not proved in the notes or in class.

Proof. Let $u = a+bi$, $v = c+di$, and $z = f+gi$ be complex numbers. We must show the Properties of Complex Arithmetic, i.e., that

(a) Complex addition is associative.

First, $u + v = (a + c) + (b + d)i$ and $(u + v) + z = ((a + c) + f) + ((b + d) + g)i$. Then, $v + z = (c + f) + (d + g)i$, so $u + (v + z) = (a + (c + f)) + (b + (d + g))i$. The result follows by the associativity of real addition.

(b) Complex addition is commutative.

We have $u + v = (a + c) + (b + d)i = (c + a) + (d + b)i = v + u$ by the commutativity of real addition.

(c) The complex additive identity is $0 = 0 + 0i$. (Example 3, p. 159)

(d) A complex additive inverse $-z$ exists. (Example 3, p. 159)

(e) Complex multiplication is associative.

By definition, $uv = (ac - bd) + (ad + bc)i$, so we have

$$(uv)w = ((ac - bd)f - (ad + bc)g) + ((ac - bd)g + (ad + bc)f)i$$

We also have $vw = (cf - dg) + (cg + df)i$ and by extension

$$\begin{aligned} u(vw) &= (a(cf - dg) - b(cg + df)) + (a(cg + df) + b(cf - dg))i \\ &= (acf - adg - bcg - bdf) + (acg + adf + bcf - bdg)i \\ &= (acf - bdf - adg - bcg) + (acg - bdg + adf + bcf)i \\ &= ((ac - bd)f - (ad + bc)g) + ((ac - bd)g + (ad + bc)f)i \\ &= (uv)w \end{aligned}$$

as desired.

(f) Complex multiplication is commutative.

Again, $uv = (ac - bd) + (ad + bc)i$ and $vu = (ca - db) + (cb + da)i$. The result follows from the commutativity of real multiplication and addition.

(g) The complex multiplicative identity is $1 = 1 + 0i$. (Example 3, p. 159)

(h) A complex multiplicative inverse z^{-1} exists iff $z \neq 0$. (Proposition 1, p. 159)

(i) Complex multiplication distributes over addition.

We have $u + v = (a + c) + (b + d)i$. Then,

$$z(u + v) = (f(a + c) - g(b + d)) + (f(b + d) + g(a + c))i$$

Now, $zu = (fa - gb) + (fb + ga)i$ and $zv = (fc - gd) + (fd + gc)i$, so by definition,

$$\begin{aligned} zu + zv &= ((fa - gb) + (fc - gd)) + ((fb + ga) + (fd + gc))i \\ &= (fa + fc - gb - gd) + (fb + fd + ga + gc)i \\ &= (f(a + c) - g(b + d)) + (f(b + d) + g(a + c))i \\ &= z(u + v) \end{aligned}$$

completing the proof. □

Recommended Problem 10.3. Let $n \in \mathbb{N}$. Prove that if $n \equiv 1 \pmod 4$, then $i^n = i$.

Proof. Let n be a natural number congruent to 1 modulo 4. Then, we may write $n = 4k + 1$ for some integer k. Notice that $i^4 = (i^2)^2 = (-1)^2 = 1$.

Therefore, $i^{4k+1} = (i^4)^k i^1 = (1)^k i = i$, as desired. □

Recommended Problem 10.4. Find all $z \in \mathbb{C}$ which satisfy

(a) $z^2 + 2z + 1 = 0$

Solution. Factor: $z^2 + 2z + 1 = (z + 1)^2$ so $z = -1 + 0i$ (by RP 10.6) □

(b) $z^2 + 2\bar{z} + 1 = 0$

Solution. Let $z = a + bi$ so $\bar{z} = a - bi$ for two real numbers a and b. Then,

$$\begin{aligned} 0 &= z^2 + 2\bar{z} + 1 \\ 0 &= (a + bi)^2 + 2(a - bi) + 1 \\ 0 &= (a^2 + 2a - b^2 + 1) + (2ab - 2b)i \end{aligned}$$

which is true if and only if both $a^2 + 2a - b^2 + 1 = 0$ and $2ab - 2b = 0$.

The second equation implies $2ab = 2b$ so $a = 1$ or $b = 0$.

If $a = 1$ then $a^2 + 2a - b^2 + 1 = 4 - b^2 = 0$, so $b = \pm 2$.

If $b = 0$, then $a^2 + 2a + 1 = (a + 1)^2 = 0$, so $a = -1$.

Therefore, the solutions are $-1 + 0i$, $1 + 2i$, and $1 - 2i$. □

(c) $z^2 = \dfrac{1 + i}{1 - i}$

Solution. Simplify: $z^2 = \frac{(1+i)^2}{2} = \frac{2i}{2} = i$. The square roots of i are $\pm(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$. □

Recommended Problem 10.5.

(a) Find all $w \in \mathbb{C}$ satisfying $w^2 = -15 + 8i$.

Solution. We rewrite $w = a + bi$ for some reals a and b. Then, $(a+bi)^2 = (a^2 - b^2) + (2ab)i = -15 + 8i$. Equating real and complex parts, $a^2 - b^2 = -15$ and $2ab = 8$.

Now, $|w^2| = |ww| = |w||w| = |w|^2$ by PM4. Then, $a^2 + b^2 = \sqrt{(-15)^2 + (8)^2} = 17$. Solving the system in a^2 and b^2, $a^2 = 1$ and $b^2 = 16$.

Therefore, $a = \pm 1$ and $b = \pm 4$. To satisfy $2ab = 8$, we must have $z = \pm(1 + 4i)$. □

(b) Find all $z \in \mathbb{C}$ satisfying $z^2 - (3 + 2i)z + 5 + i = 0$.

Solution. We apply the quadratic formula. The discriminant is a solution to $w^2 = (3+2i)^2 - 4(1)(5+i) = (5+12i) - (20+4i) = -15+8i$. From above, a solution is $w = 1+4i$. Therefore, the solutions are $z = \frac{(3+2i) \pm (1+4i)}{2(1)}$.

The first is $z = \frac{(3+2i)+(1+4i)}{2} = 2 + 3i$ and the second is $z = \frac{(3+2i)-(1+4i)}{2} = 1 - i$. □

Recommended Problem 10.6. Let $z, w \in \mathbb{C}$. Prove that if $zw = 0$ then $z = 0$ or $w = 0$.

Proof. Let z and w be complex numbers such that $zw = 0$. Suppose for a contradiction that both z and w are non-zero. Then, by PM1, $|z| \neq 0$ and $|w| \neq 0$. However, by PM4, $|zw| = |z||w| \neq 0$, which is a contradiction, since $zw = 0$.

Therefore, z or w is zero. □

Recommended Problem 10.7. Let $a, b, c \in \mathbb{C}$. Prove: if $|a| = |b| = |c| = 1$, then $\overline{a + b + c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Proof. First, consider some arbitrary complex number $z = a + bi$ with modulus 1. By definition, $a^2 + b^2 = 1^2 = 1$. Then, $z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{1} = a - bi = \bar{z}$

Let a, b, and c be complex numbers with modulus 1. From above, $a^{-1} = \bar{a}$, $b^{-1} = \bar{b}$, and $c^{-1} = \bar{c}$. The conclusion immediately follows from PCJ2:

$$\begin{aligned} \overline{a + b + c} &= \bar{a} + \bar{b} + \bar{c} \\ &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \end{aligned}$$

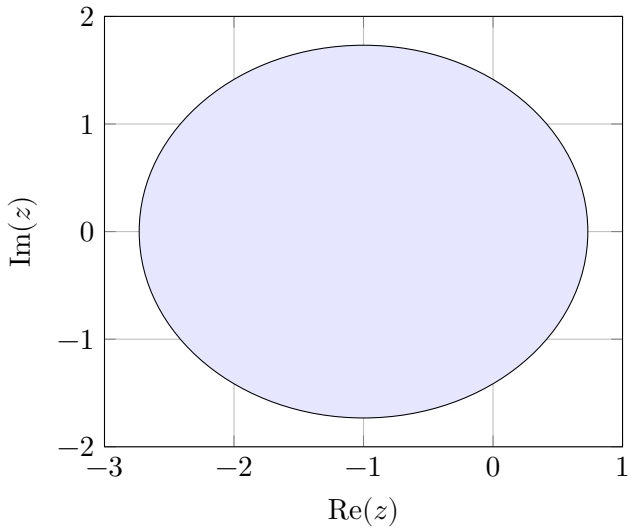
□

Recommended Problem 10.8. Find all $z \in \mathbb{C}$ satisfying $z^2 = |z|^2$.

Proof. Let z be a complex number. Recall that $|z|^2 = \bar{z}z$ by PM3. Then, we have $z^2 = \bar{z}z$ so $z = \bar{z}$, that is, $z - \bar{z} = 0$. By PCJ3, this is true if $2\,\mathrm{Im}(z)i = 0$, which means that z is purely real. Therefore, z is any purely real number. □

Recommended Problem 10.9. Find all $z \in \mathbb{C}$ satisfying $|z+1|^2 \leq 3$ and shade the corresponding region in the complex plane.

Solution. We write $z = a + bi$, so $|z+1|^2 = |(a+1)+bi|^2 = (\sqrt{(a+1)^2+b^2})^2 = (a+1)^2 + b^2$. Then, we are shading the inside of the circle defined by $(a+1)^2 + b^2 = 3$.



This is the circle centered at $(-1, 0)$ with radius $\sqrt{3}$. □

Recommended Problem 10.10. Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that if $|z| = 1$ or $|w| = 1$, then $\left|\dfrac{z-w}{1-\overline{z}w}\right| = 1$.

Proof (by sooshi). Let z and w be complex numbers such that $\overline{z}w \neq 1$. Suppose that $|z| = 1$ or $|w| = 1$. If $z = w$ and $|z| = |w| = 1$, then $\overline{z}w = \overline{z}z = |z|^2 = 1$. Therefore, $z \neq w$.

Now, consider the case when $|z| = 1$. Then,

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \frac{|z-w|}{|1-\overline{z}w|} = \frac{|z||z-w|}{|z||1-\overline{z}w|} = \frac{(1)|z-w|}{|z-z\overline{z}w|} = \frac{|z-w|}{|z-w|} = 1$$

Likewise, if $|w| = 1$, then

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \frac{|z-w|}{|1-\overline{z}w|} = \frac{|z-w|}{|w\overline{w}-\overline{z}w|} = \frac{|z-w|}{|w||\overline{w}-\overline{z}|} = \frac{|z-w|}{|w-z|} = 1$$

since $|w-z| = |-(z-w)| = |-1||z-w| = |z-w|$, completing the proof. □

Recommended Problem 10.11. Show that for all complex numbers z, $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$.

Proof. Let $z = r\operatorname{cis}\theta$ be a complex number. Then, $|z| = r$, $\operatorname{Re}(z) = r\cos\theta$ and $\operatorname{Im}(z) = r\sin\theta$. Due to the symmetry of sine and cosine, instead of taking absolute values, we restrict without loss of generality to the first quadrant $0 \leq \theta \leq \frac{\pi}{2}$. Now,

$$\begin{aligned}
\operatorname{Re}(z) + \operatorname{Im}(z) &= r(\cos\theta + \sin\theta) \\
&= r\sqrt{2}\frac{\sqrt{2}}{2}(\cos\theta + \sin\theta) \\
&= r\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right) \\
&= r\sqrt{2}\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right) \\
&= r\sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \\
&\leq r\sqrt{2}(1) \\
&= \sqrt{2}|z|
\end{aligned}$$

completing the proof. □

Recommended Problem 10.12. Use *De Moivre's Theorem* (DMT) to prove that $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$ for all $\theta \in \mathbb{R}$.

Proof. Let $\theta \in \mathbb{R}$ and note that by DMT, we have

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

so we may say that $\sin 4\theta = \operatorname{Im}((\cos\theta + i\sin\theta)^4)$. Expanding this quantity by hand,

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= (\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta)^2 \\
&= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta + 4i\cos^3\theta\sin\theta - 4i\sin^3\theta\cos\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + (4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta)i
\end{aligned}$$

and we have that

$$\sin 4\theta = \operatorname{Im}((\cos\theta + i\sin\theta)^4) = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$$

as desired. □

Recommended Problem 10.13. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Show that $z = (a+bi)^n + (a-bi)^n$ is real.

Proof. Let n be a natural number and $u = a + bi$ be a complex number. Then, $\overline{u} = a - bi$. It inductively follows from PCJ4 and the associativity of multiplication that $(\overline{u})^n = \overline{u^n}$.

Now, the fact that $z = u^n + \overline{u^n}$ is real follows immediately from PCJ3. □

Recommended Problem 10.14. An *n-th root of unity* is any complex solution to $z^n = 1$. Prove that if w is an n-th root of unity, $\frac{1}{w}$ is also an n-th root of unity.

Proof. Let n be a natural number and w be an n-th root of unity, so $w^n = 1$. Knowing that $1 = \operatorname{cis} 0$, the CNRT states that $w = \operatorname{cis}(\frac{2k\pi}{n})$ for some $0 \leq k < n$.

By PM$\mathbb{C}$, notice that $w \operatorname{cis}(-\frac{2k\pi}{n}) = \operatorname{cis}(\frac{2k\pi}{n} - \frac{2k\pi}{n}) = \operatorname{cis} 0 = 1$, so $\operatorname{cis}(-\frac{2k\pi}{n})$ is the multiplicative inverse w^{-1} of w. Now, since cis is 2π-periodic, we have

$$\operatorname{cis}\left(-\frac{2k\pi}{n}\right) = \operatorname{cis}\left(2\pi - \frac{2k\pi}{n}\right) = \operatorname{cis}\left(\frac{2n\pi - 2k\pi}{n}\right) = \operatorname{cis}\left(\frac{2(n-k)\pi}{n}\right)$$

but since $0 \leq k < n$, we also have that $0 \leq n - k < n$. Therefore, by the CNRT, w^{-1} is an n-th root of unity. □

Recommended Problem 10.15. A complex number z is called a *primitive* n-th root of unity if $z^n = 1$ and $z^k \neq 1$ for all $1 \leq k \leq n - 1$.

(a) For each $n = 1, 3, 5, 6$ list all the primitive n-th roots of unity.

Solution. Recall that $1^x = 1$ for any real x. Applying the CNRT, there are n n-th roots of unity, of the form

$$z = \operatorname{cis}\left(\frac{2\pi k}{n}\right)$$

for some integer $0 \leq k < n$. Note that 1 is always an n-th root of unity but only a primitive first root of unity. Therefore, we can ignore the case $k = 0$.

The only primitive 1st root of unity is 1.

The primitive 3rd roots of unity are $\operatorname{cis}\frac{2\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ and $\operatorname{cis}\frac{4\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

For this, we remain in polar form as calculating sines and cosines of fractions over 5 is *pain*. The primitive 5th roots of unity are $\operatorname{cis} 0 = 1$, $\operatorname{cis}\frac{2\pi}{5}$, $\operatorname{cis}\frac{4\pi}{5}$, $\operatorname{cis}\frac{6\pi}{5}$, and $\operatorname{cis}\frac{8\pi}{5}$.

The 6th roots of unity are $\operatorname{cis}\frac{2\pi k}{6} = \operatorname{cis}\frac{\pi k}{3}$. However, when $k = 2$, $k = 3$, and $k = 4$, these are also 2nd/3rd roots of unity. Thus, the primitive roots of unity are $\operatorname{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\operatorname{cis}\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. □

(b) Let z be a primitive n-th root of unity. Prove the following statements:

i. For any $j \in \mathbb{Z}$, $z^j = 1$ if and only if $n \mid j$.

Proof. Let n be a natural number, j be an integer, and z be a primitive n-th root of unity so $z^n = 1$. Proceed by mutual implication.

($\Rightarrow$) Suppose $z^j = 1$. By the Division Algorithm, $j = qn + r$ for integers q and $0 \leq r < n$. Then, $1 = z^j = z^{qn+r} = z^{qn}z^r = (z^n)^q z^r = 1^q z^r = z^r$.

If $r = 0$, then $j = qn$ and $n \mid j$. Otherwise, we have $1 \leq r \leq n-1$ and $z^r = 1$, which is a contradiction to the fact that z is a primitive n-th root of unity.

Therefore, $r = 0$ and $n \mid j$.

($\Leftarrow$) If $n \mid j$ and $j = nk$ for an integer k, then $z^j = z^{nk} = (z^n)^k = 1^k = 1$. □

ii. For any $m \in \mathbb{Z}$, if $\gcd(m, n) = 1$, then z^m is a primitive n-th root of unity.

Proof (new and improved by sooshi). Let z be a primitive n-th root of unity and m an integer coprime to n.

Suppose for a contradiction that z^m is a k-th root of unity for some $1 \leq k < n$. Then, $(z^m)^k = z^{mk} = 1$. From above, this implies that $n \mid mk$ and by CAD, $n \mid k$. However, BBD gives that $n \leq k$, which is a contradiction.

Therefore, z^m is a primitive n-th root of unity. □

Recommended Problem 10.16. Let u and v be fixed complex numbers. Let ω be a non-real cube root of unity. For each $k \in \mathbb{Z}$, define $y_k \in \mathbb{C}$ by the formula

$$y_k = \omega^k u + \omega^{-k} v$$

(a) Compute y_1, y_2, and y_3 in terms of u, v, and ω.

Solution. From RP15(a), the only real cube root of unity is 1, so $\omega \neq 1$. In fact, $\omega = \operatorname{cis} \frac{n\pi}{3}$ for either $n = 2$ or $n = 4$.

If $n = 2$, then $\omega^{-1} = \operatorname{cis} \frac{-2\pi}{3} = \operatorname{cis} \frac{4\pi}{3}$. If $n = 4$, then $\omega^{-1} = \operatorname{cis} \frac{-4\pi}{3} = \operatorname{cis} \frac{2\pi}{3}$.

However, using the standard form from RP15(a), $\operatorname{cis} \frac{2\pi}{3} = \overline{\operatorname{cis} \frac{4\pi}{3}}$. Therefore, $\omega^{-1} = \overline{\omega}$.

Now, $y_1 = \omega u + \overline{\omega} v$, $y_2 = \omega^2 u + \overline{\omega}^2 v$, and $y_3 = \omega^3 u + \overline{\omega}^3 v = u + v$. □

(b) Show that $y_k = y_{k+3}$ for any $k \in \mathbb{Z}$.

Proof. Let k be an integer. Then, knowing that both ω and $\overline{\omega}$ are cube roots of unity,

$$\begin{aligned} y_{k+3} &= \omega^{k+3} u + \overline{\omega}^{k+3} v \\ &= \omega^k \omega^3 u + \overline{\omega}^k \overline{\omega}^3 v \\ &= \omega^k u + \overline{\omega}^k v \\ &= y_k \end{aligned}$$

completing the proof. □

(c) Show that for any $k \in \mathbb{Z}$,

$$y_k - y_{k+1} = \omega^k (1 - \omega)(u - \omega^{k-1} v)$$

Proof. Let k be an integer. Expand the right-hand side:

$$\begin{aligned}\omega^k(1-\omega)(u-\omega^{k-1}v) &= (\omega^k-\omega^{k+1})(u-\omega^{k-1}v)\\ &= \omega^k u-\omega^{2k+1}v-\omega^{k+1}u+\omega^{2k+2}v\\ &= (\omega^k u+\omega^{2k+2}v)-(\omega^{k+1}u+\omega^{2k+1}v)\end{aligned}$$

To simplify, we show that $\omega^{2k+2}=\omega^{-k}$. Equivalently, $\omega^{2k+2}\omega^k=\omega^{3k+2}=1$. Let $j=k+1$. Then,

$$\omega^{3k+2}=\omega^{3(j-1)+2}=\omega^{3j-1}=(\omega^3)^j\omega^{-1}=1^j\omega^{-1}=\omega^{-1}$$

as desired. Now, we have $\omega^{2k+2}=\omega^{-k}$ and $\omega^{2k+1}=\omega^{-(k+1)}$ so

$$\begin{aligned}\omega^k(1-\omega)(u-\omega^{k-1}v) &= (\omega^k u+\omega^{2k+2}v)-(\omega^{k+1}u+\omega^{2k+1}v)\\ &= (\omega^k u+\omega^{-k}v)-(\omega^{k+1}u+\omega^{-(k+1)}v)\\ &= y_k-y_{k+1}\end{aligned}$$

□

10.3 Challenges

Challenge 10.1. Let $z,w\in\mathbb{C}$.

(a) Prove that $|z+w|\leq|z|+|w|$.

Proof. This is the Triangle Inequality, for which a geometric proof is provided in Chapter 10.3. In short, for complex numbers $z=a+bi$ and $w=c+di$, we consider a triangle $\triangle OZW$ with points $O(0,0)$, $Z(a,b)$, and $W(c,d)$ in the complex plane. Then, $|z|=\ell_{OZ}$, $|w|=\ell_{OW}$, and $|z+w|=\ell_{ZW}$. The length of one side of a triangle cannot exceed the sum of the lengths of the other two sides.

Equivalently, $\ell_{ZW}\leq\ell_{OZ}+\ell_{OW}$. □

(b) Prove that $||z|-|w||\leq|z-w|\leq|z|+|w|$.

Proof. Let z and w be complex numbers. We prove the inequalities separately.

We apply the Triangle Inequality with z and $-w$. Then, $|z+(-w)|\leq|z|+|-w|$ but $|-w|=|-1||w|=|w|$ by PM4, so we have $|z-w|\leq|z|+|w|$.

Now, notice that $|z|=|(z-w)+w|\leq|z-w|+|w|$ so $|z|-|w|\leq|z-w|$.

Likewise, $|w|=|(w-z)+z|\leq|w-z|+|z|$ so $|z|-|w|\geq-|w-z|$.

Like the absolute value in $\mathbb{R}$, we have by PM4 $|w-z|=|-1||z-w|=1|z-w|=|z-w|$, so if we combine the above two inequalities, we have $||z|-|w||\leq|z-w|$.

Equivalently, using the same triangle from above, this follows from the fact that any one side of a triangle is longer than the difference of the other two sides. □

Challenge 10.2. Let $a, b, c \in \mathbb{C}$. Show that if $\dfrac{b-a}{a-c} = \dfrac{a-c}{c-b}$ then $|b-a| = |a-c| = |c-b|$.

Challenge 10.3. Let $n \geq 2$ be an integer. Prove that

$$\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 = \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right)$$

Proof (with help from Ainsley, Kenson, Mabel). Let $n \neq 1$ be a natural number. Then, we have that the n-th roots of unity are given by

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

for $k = 0, 1, 2, \ldots, n-1$. Let z be the sum of the n-th roots of unity. Then,

$$z = \sum_{k=0}^{n-1} \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

The conclusion can equivalently be stated as that $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) = 0$. The only complex number that satisfies this is $z = 0$.

Now, let $a = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, the root of unity with $k = 1$. Then, we have that each root of unity is given by a^j for $j = 1, 2, \ldots, n$. Since $n \neq 1$, $a = \operatorname{cis}\frac{2\pi}{n} \neq 1$ and $z = 1 + a + a^2 + \cdots + a^{n-1}$.

Recall that the polynomial $a^n - 1$ for $n \geq 2$ factors as $(a-1)(a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1)$. It follows that $a^n - 1 = 1 - 1 = 0$ and $0 = (a-1)z$ so, from above, $a \neq 1$ so $z = 0$. □

Chapter 11

Polynomials

11.1 Warm-Up Exercises

Warm-Up Exercise 11.1. Find a real cubic polynomial whose roots include 1 and i.

Solution. Apply the Factor Thorem to create $f(x) = (x-1)(x-i)(x-r)$. To ensure the polynomial is real, make $(x-r)$ the conjugate of $(x-i)$, i.e., $r = -i$. Then, $f(x) = (x-1)(x^2+1) = x^3 - x^2 + x - 1$. □

Warm-Up Exercise 11.2. Divide $f(x) = x^3 + x^2 + x + 1$ by $g(x) = x^2 + 4x + 3$ to find the quotient $q(x)$ and remainder $r(x)$ that satisfy the requirements of the *Division Algorithm for Polynomials* (DAP)

Solution. Perform polynomial long division:

$$\begin{array}{rr|l} & & x - 3 \\ \hline x^2+4x+3) & x^3 + x^2 + x + 1 & \\ & -x^3 - 4x^2 - 3x & \\ \hline & -3x^2 - 2x + 1 & \\ & 3x^2 + 12x + 9 & \\ \hline & 10x + 10 & \end{array}$$

and conclude that $q(x) = 10x + 10$ and $r(x) = x - 3$. □

11.2 Recommended Problems

Recommended Problem 11.1. Let $z \in \mathbb{C}$. Prove that $(x-z)(x-\overline{z}) \in \mathbb{R}[x]$.

Proof. Let z be a complex number. Expand the product to obtain

$$\begin{aligned}(x-z)(x-\overline{z}) &= x^2 - zx - \overline{z}x + z\overline{z} \\ &= x^2 - (z+\overline{z})x + z\overline{z}\end{aligned}$$

which is a polynomial in x with coefficients 1, $-(z+\overline{z})$, and $z\overline{z}$. Clearly, $1 \in \mathbb{R}$. From PCJ3, we have $z + \overline{z} = 2\operatorname{Re} z$ so $-(z+\overline{z}) = -2\operatorname{Re} z \in \mathbb{R}$. Also, from PM3, $z\overline{z} = |z|^2 \in \mathbb{R}$. Therefore, the polynomial is a member of $\mathbb{R}[x]$. □

Recommended Problem 11.2. Prove that there exists a polynomial in $\mathbb{Q}[x]$ with the root $2 - \sqrt{7}$.

Proof. We propose $f(x) = x^2 - 4x - 3 \in \mathbb{Q}[x]$.

$$f(2-\sqrt{7}) = (2-\sqrt{7})^2 - 4(2-\sqrt{7}) - 3 = 11 - 4\sqrt{7} - 8 + 4\sqrt{7} - 3 = 0$$ □

Recommended Problem 11.3. For each of the following polynomials $f(x) \in \mathbb{F}[x]$, write $f(x)$ as a product of irreducible polynomials in $\mathbb{F}[x]$.

(a) $x^2 - 2x + 2 \in \mathbb{C}[x]$

Solution. We apply the quadratic formula to find that $x = \frac{2+\sqrt{-4}}{2} = 1 + i$. Then, we also have $x = 1 - i$ as a solution. Therefore, we may write in irreducible polynomials $f(x) = (x - 1 - i)(x - 1 + i)$. □

(b) $x^2 + (-3i + 2)x - 6i \in \mathbb{C}[x]$

Solution. By inspection, $x = -2$ is a root. Divide by $g(x) = x + 2$ to obtain $q(x) = x - 3i$. Therefore, we write in irreducible polynomials $f(x) = (x + 2)(x - 3i)$. □

(c) $2x^3 - 3x^2 + 2x + 2 \in \mathbb{R}[x]$

Solution. The RRT gives $x = 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}$ as candidates for roots of f. We find that $f(-\frac{1}{2}) = 0$, so we divide by $g(x) = 2x + 1$ to find $q(x) = x^2 - 2x + 2$. Now, the discriminant of q is negative, so it has no real solutions and is irreducible in $\mathbb{R}[x]$. Therefore, we write $f(x) = (2x + 1)(x^2 - 2x + 2)$. □

(d) $3x^4 + 13x^3 + 16x^2 + 7x + 1 \in \mathbb{R}[x]$

Solution. By inspection, $x = -1$ is a root. Divide by $g(x) = x + 1$ to obtain $q(x) = 3x^3 + 10x^2 + 6x + 1$. To find roots of this cubic, the RRT gives candidates $x = 1, -1, \frac{1}{3}, -\frac{1}{3}$. In fact, $q(-\frac{1}{3}) = 0$. Dividing $q(x)$ by $(3x + 1)$, we obtain the factor $(x^2 + 3x + 1)$. The discriminant of this quadratic is positive and it has roots $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$. Therefore, $f(x) = (x + 1)(3x + 1)(x - \frac{3}{2} + \frac{\sqrt{5}}{2})(x - \frac{3}{2} - \frac{\sqrt{5}}{2})$. □

(e) $x^4 + 27x \in \mathbb{C}[x]$

Solution. Factor: $f(x) = x(x^3 + 27)$. The roots are $x = 0$ and $x = \sqrt[3]{-27} = 3\sqrt[3]{-1}$. By the CNRT, the cube roots of -1 are -1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Therefore,

$$f(x) = x(x+3)\left(x - \frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)\left(x - \frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)$$ □

Recommended Problem 11.4. Let $g(x) = x^3 + bx^2 + cx + d \in \mathbb{C}[x]$ be a monic cubic polynomial. Let z_1, z_2, and z_3 be three roots of $g(x)$ such that

$$g(x) = (x - z_1)(x - z_2)(x - z_3)$$

Prove that

$$\begin{aligned} z_1 + z_2 + z_3 &= -b \\ z_1z_2 + z_2z_3 + z_3z_1 &= c \\ z_1z_2z_3 &= -d \end{aligned}$$

Proof. Let g be a monic cubic polynomial over $\mathbb{C}$, where z_1, z_2, and z_3 are its roots. Then, by CPN, $g(x) = x^3 + bx^2 + cx + d = (x - z_1)(x - z_2)(x - z_3)$ for some coefficients $b, c, d \in \mathbb{C}$. We expand using standard arithmetic:

$$\begin{aligned} x^3 + bx^2 + cx + d &= (x - z_1)(x - z_2)(x - z_3) \\ &= (x^2 - xz_1 - xz_2 + z_1z_2)(x - z_3) \\ &= x^3 - x^2z_1 - x^2z_2 + z_1z_2x - x^2z_3 - z_1z_3x - z_2z_3x - z_1z_2z_3 \\ &= x^3 - (z_1 + z_2 + z_3)x^2 + (z_1z_2 + z_2z_3 + z_3z_1)x - z_1z_2z_3 \end{aligned}$$

Recall that two polynomials are defined to be equal if and only if their coefficients agree. Therefore, $b = -(z_1 + z_2 + z_3)$, $c = z_1z_2 + z_2z_3 + z_3z_1$, and $d = -z_1z_2z_3$ and the conclusion immediately follows. □

Recommended Problem 11.5. Using the Rational Roots Theorem, prove that $\sqrt{3} + \sqrt{7}$ is irrational.

Proof. Let $a = \sqrt{3} + \sqrt{7}$. Then, $a^2 = 10 + 2\sqrt{21}$ and $a^2 - 10 = 2\sqrt{21}$. Squaring again, $a^4 - 20a^2 + 100 = 84$, i.e., $a^4 + 20a^2 - 16 = 0$.

Now, we can let $f(x) = x^4 - 20x^2 + 16$ such that $f(a) = 0$. The RRT gives that rational roots of f are of the form p/q with coprime integers p and q where $p \mid 16$ and $q \mid 1$. The divisors of 1 are ± 1 and of 16 are $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$. Note that f is even, so we need only test $x = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

Now, $f(1) = 5$, $f(\frac{1}{2}) = -\frac{175}{16}$, $f(\frac{1}{4}) = -\frac{3775}{256}$, $f(\frac{1}{8}) = -\frac{64255}{4096}$, and $f(\frac{1}{16}) = -\frac{1043455}{65536}$.

Therefore, f has no rational roots. However, a is a root of f, therefore, a is irrational. □

Recommended Problem 11.6.

(a) Prove that for every prime p, there exists a polynomial $f(x)$ over $\mathbb{Z}_p$, of degree p, such that every element of $\mathbb{Z}_p$ is a root of $f(x)$.

Proof. Let p be a prime number. Then, $\mathbb{Z}_p$ is a field. For each element $[n] \in \mathbb{Z}_p$, there is a linear factor $([1]x - [n]) \in \mathbb{Z}_p[x]$. The product of polynomials is well-defined and is a polynomial, so we may say that the polynomial $f(x) \in \mathbb{Z}_p[x]$

$$f(x) = \prod_{[i]\in\mathbb{Z}_p} ([1]x - [i])$$

has p roots corresponding to each of the p elements in $\mathbb{Z}_p$. The degree of a product is the sum of the degrees of the factors, but each factor is linear with degree 1 so the sum is simply p. □

(b) Prove that for every prime p, there exists a polynomial $f(x)$ over $\mathbb{Z}_p$, of degree p, which has no roots in $\mathbb{Z}_p$.

Proof. Let p be a prime number and let $g(x)$ be the polynomial from (a) above for p. Then, $g(x) \equiv 0 \pmod{p}$ for any $x \in \mathbb{Z}_p$. Therefore, $g(x) \not\equiv 1 \pmod{p}$ for any x and we may say the polynomial $f(x) = g(x) - 1$ has no solutions in $\mathbb{Z}_p$. □

Recommended Problem 11.7. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ with degree n. We say $f(x)$ is *palindromic* if the coefficients a_j satisfy

$$a_{n-j} = a_j \quad \text{for all} \quad 0 \leq j \leq n$$

Prove that

(a) If $f(x)$ is a palindromic polynomial and $c \in \mathbb{C}$ is a root of $f(x)$, then c must be non-zero, and $\frac{1}{c}$ is also a root of $f(x)$.

Proof. Let $f(x) \in \mathbb{C}[x]$ be a palindromic polynomial with coefficients a_n and root c so

$$0 = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$$

Since $f(x)$ has degree n, $a_n \neq 0$. As $f(x)$ is palindromic, $a_0 \neq 0$. Suppose that $c = 0$ and substitute above. We have that $a_0 = 0$, which is a contradiction. Therefore, $c \neq 0$. Now, multiplying through by c^{-n}, we have

$$0 = a_n + a_{n-1}c^{-1} + \cdots + a_1 c^{-n+1} + a_0 c^{-n}$$

but since $f(x)$ is palindromic we substitute a_{n-j} for a_j and write

$$0 = a_0 + a_1\left(\frac{1}{c}\right) + \cdots + a_{n-1}\left(\frac{1}{c}\right)^{n-1} + a_n\left(\frac{1}{c}\right)^n$$

But this is just saying $f(\frac{1}{c}) = 0$, that is, $\frac{1}{c}$ is a root of $f(x)$. □

(b) If $f(x)$ is a palindromic polynomial of odd degree, then $f(-1) = 0$.

Proof. Let $f(x)$ be a palindromic polynomial in $\mathbb{C}$ with odd degree n and coefficients a_n. Since n is odd, we have $n = 2k + 1$ for some integer k. Then,

$$f(-1) = a_{2k+1}(-1)^{2k+1} + a_{2k}(-1)^{2k} + \cdots + a_1(-1) + a_0$$

and we apply the fact that $a_{n-j} = a_j$ for all $0 \leq j \leq k$ to get

$$f(-1) = a_0(-1)^{2k+1} + a_1(-1)^{2k} + \cdots + a_k(-1)^{k+1} + a_k(-1)^k + \cdots + a_1(-1) + a_0$$

Notice that there are an even $(n + 1 = 2k + 2)$ number of terms. We pair them by common coefficients. Let $0 \leq i \leq k$. Then, the coefficient a_i appears in the terms $a_i(-1)^{2k+1-i}$ and $a_i(-1)^i$. The difference in the powers is $2(k - i) + 1$, an odd number. Therefore, one is even and the other is odd. Suppose WLOG that i is even. Then, $a_i(-1)^{2k+1-i} = -a_i$ and $a_i(-1)^i = a_i$.

It follows that each term cancels its palindromic term, and the resulting sum is 0. □

(c) If $\deg f = 1$ and $f(x)$ is a monic, palindromic polynomial, then $f(x) = x + 1$.

Proof. Let $f(x)$ be a first-degree polynomial in $\mathbb{C}$, that is, $f(x) = a_1x + a_0$. Since $f(x)$ is monic, its leading coefficient a_1 is 1. However, since $f(x)$ is palindromic, $a_{\deg f-1} == a_{1-1} = a_0 = 1$ as well. Therefore, $f(x) = x + 1$. □

11.3 Challenge

Challenge 11.1. We call a polynomial primitive if the greatest common divisor of all of its coefficients is 1. Show that the product of two primitive polynomials is again primitive.