MATH 137 Fall 2020: Practice Assignment 1

Q01. Solve the following equations/inequalities:

1. $|x - 2| = 5 \iff x - 2 = \pm 5 \iff x = 2 \pm 5 \iff x = -3, 7$

2. $|x - 4| = |3x + 2|$: Take cases: $x - 4 = 3x + 2 \iff x = -3$ and $x - 4 = -3x - 2 \iff x = \frac{1}{2}$. Check: $|-3 - 4| = |-9 + 2|$ and $|\frac{1}{2} - 4| = \frac{7}{2} = |\frac{3}{2} + 2|$, so $x = -3, \frac{1}{2}$.

3. $|x + 1| > 1 =$ "distance between x and -1 greater than 1" $= x \in \mathbb{R} \setminus [-2, 0] = (-\infty, -2) \cup (0, \infty)$

4. $|x + 3| + |1 - 2x| \leq 5$. Take cases $x \in (-\infty, -3], [-3, \frac{1}{2}], [\frac{1}{2}, \infty)$:

   1. $-(x + 3) + (1 - 2x) \leq 5 \iff x \geq -\frac{3}{7}$ but $x \in (-\infty, -3]$ so no solution
   2. $(x + 3) + (1 - 2x) \leq 5 \iff x \geq -1$ but $x \in [-3, \frac{1}{2}]$ so $x \in [-1, \frac{1}{2}]$
   3. $(x + 3) - (1 - 2x) \leq 5 \iff x \leq 1$ but $x \in [\frac{1}{2}, \infty)$ so $x \in [\frac{1}{2}, 1]$

   Therefore $x \in [-1, \frac{1}{2}] \cup [\frac{1}{2}, 1] = [-1, 1]$.

5. $|x - 4||x + 2| = |(x - 4)(x + 2)| = |x^2 - 2x - 8| > 7$. Take cases:

   1. $x^2 - 2x - 8 > 7 \iff x^2 - 2x - 15 > 0 \iff (x - 5)(x + 3) > 0 \iff x \in (-\infty, -3) \cup (5, \infty)$
   2. $x^2 - 2x - 8 < -7 \iff x^2 - 2x - 1 < 0 \iff (x - 1)^2 < 0$ which is false for all real x.

   Therefore, $x \in (-\infty, -3) \cup (5-, \infty)$.

Q02. Prove that, for any real numbers a and b:

1. $|a| - |b| \leq |a + b|$

   Proof. Moving the $|b|$ term to the other side, $|a| \leq |a + b| + |b|$. Let $b' = -b$, so $|a| \leq |a + (-b')| + |-b'|$. Add some zeroes, $|a - 0| \leq |a - b'| + |b' - 0|$, which holds by the triangle inequality. $\qquad\square$

2. $|a| + |b| \leq |a + b|$

   Proof. Again move the $|b|$ term to the other side, $|a| \leq |a - b| + |b|$, and add some zeroes, $|a - 0| \leq |a - b| + |b - 0|$, which holds by the triangle inequality. $\qquad\square$

3. $|a - b| \leq |a + b|$ or $|a + b| \leq |a - b|$

   Proof. Consider for $a = b \neq 0$: the first is true as $0 \leq 2a$ and the second is false as $2a \not\leq 0$. Since the second is false for some case, it is false generally.

   For $a = -b \neq 0$: the first is now false as $2a \not\leq 0$.

   Therefore, both statements are false for some $x \in \mathbb{R}$, so they are not true. $\qquad\square$

Q03. Write the expression

$$\frac{1}{2}(a + b + |a - b|)$$

as a piecewise function. You should consider 2 cases. What is this expression calculating for any two real numbers a and b? What about the expression

$$\frac{1}{2}(a + b - |a - b|)?$$

Solution. Consider the cases:

$$\frac{1}{2}(a + b + |a - b|) = \begin{cases} \frac{1}{2}(a + b + a - b) & a \geq b \\ \frac{1}{2}(a + b - a + b) & a < b \end{cases}$$

$$= \begin{cases} a & a \geq b \\ b & a < b \end{cases}$$

Which is just $\max\{a, b\}$.

Again, consider the cases:

$$\frac{1}{2}(a + b - |a - b|) = \begin{cases} \frac{1}{2}(a + b - a + b) & a \geq b \\ \frac{1}{2}(a + b + a - b) & a < b \end{cases}$$

$$= \begin{cases} b & a \geq b \\ a & a < b \end{cases}$$

Which is just $\min\{a, b\}$. $\square$