MATH 137 Fall 2020: Practice Assignment 3

Q01. Assuming $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ where $a_n \geq 0$ and $b_n \geq 0$, determine

$$\lim_{n\to\infty} (a_n \sin(n) + b_n \cos(n))$$

Proof. First, notice that given a convergent sequence $\{k_n\}$ with limit L, by the product and constant arithmetic rules for limits of sequences, $\{-k_n\}$ has limit $-L$. Therefore, $\lim_{n\to\infty} -a_n = \lim_{n\to\infty} -b_n = 0$.

Now, consider the term $a_n \sin n$. Because $-1 \leq \sin n \leq 0$ and $a_n \geq 0$, $-a_n \leq a_n \sin n \leq a_n$. By the squeeze theorem, $a_n \sin n \to 0$.

Applying the same argument to $\{b_n\}$, we find $b_n \cos n \to 0$.

By the sum rule, $\lim_{n\to\infty} (a_n \sin(n) + b_n \cos(n)) = 0 + 0 = 0$. □

Q02. Let's examine how absolute values and limits interact.

(a) The statement

$$\text{If } \lim_{n\to\infty} |a_n| = |L| \text{ then } \lim_{n\to\infty} a_n = L.$$

is false in general. Provide a counter-example.

Proof. Let $a_n = 1$ if n is even, and $a_n = -1$ otherwise. $|a_n|$ is constant for all n, so the limit is that constant, i.e. 1. However, a_n clearly has no limit. □

(b) The statement

$$\text{If } \lim_{n\to\infty} a_n = L \text{ then } \lim_{n\to\infty} |a_n| = |L|.$$

is true. Show this using the definition of limits.

Hint: $||a| - |b|| \leq |a - b|$. (Even though it is not necessary for this question, you should be able to show that the hint is true.)

Proof. Let $\epsilon > 0$. There is then an N such that $n \geq N$ implies $|a_n - L| > \epsilon$. We must find an N so $n \geq N$ implies $||a_n| - |L|| > \epsilon$. Consider the same N and the hint:

$$||a_n| - |L|| \leq |a_n - L| < \epsilon$$ □

(c) Is the statement

$$\text{If } \lim_{n\to\infty} |a_n| = 0 \text{ then } \lim_{n\to\infty} a_n = 0.$$

true? If so, argue why, if not, provide a counterexample.

Proof. Let $\epsilon > 0$. There is then an N such that $n \geq N$ implies $||a_n| - 0|| < \epsilon$. However, $||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0|$. This means that $n \geq N \implies |a_n - 0| < \epsilon$, which is precisely what must be shown to prove that $a_n \to 0$. □

Q03. Compute the following limits using any method.

(a) $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{n^2}$

Proof. First, consider the limit of $\frac{1}{n^2}$. This is trivially zero (let $N = \epsilon^{-1/2}$). By the constant multiple rule, $-\frac{1}{n^2} \to 0$ as well.

Now, recall that $-1 \leq \sin n^2 \leq 1$ for all n. Since $\frac{1}{n^2} \geq 0$, we can also say that $-\frac{1}{n^2} \leq \frac{\sin n^2}{n^2} \leq \frac{1}{n^2}$ for all n. By the squeeze theorem, the limit is 0. □

(b) $\lim\limits_{n\to\infty} \dfrac{3n-(-1)^n}{n}$

Proof. Let $a_n = \frac{3n-(-1)^n}{n}$.

When n is odd, $(-1)^n = -1$, so $a_n = \frac{3n+1}{n}$. Otherwise, $(-1)^n = 1$, so $a_n = \frac{3n-1}{n}$.

The limits of both of these are 3, since $\frac{3n\pm1}{n} = 3 \pm \frac{1}{n}$, and $\frac{1}{n}$ converges to zero.

Notice that $\frac{3n+1}{n} > \frac{3n-1}{n}$ for all n. Therefore, $\frac{3n+1}{n} \geq a_n \geq \frac{3n-1}{n}$ for all n. By the squeeze theorem, the limit is 3. □

(c) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$

Hint: Write $n! = 1 \cdot 2 \cdot 3 \ldots n$ and $n^n = n \cdot n \cdot n \ldots n$ and use the fact that $0 < \frac{n!}{n^n}$

Proof. Let $a_n = \frac{n!}{n^n}$. Expand using the definitions of the factorial and exponentiation:

$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\ldots n}{n\cdot n\cdot n\ldots n} = \frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n-1}{n}\cdot\frac{\cancel{n}}{\cancel{n}} = \frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n-1}{n} = \prod_{k=1}^{n-1}\frac{k}{n}$$

For each term in the product, since k is constant with respect to n, $\frac{k}{n} \to k\frac{1}{n} \to 0$. By the product rule for limits, $a_n \to \prod 0 = 0$. □

(d) $\lim\limits_{n\to\infty} \dfrac{3n^3+2n^2-n-1}{n^3+n+3}$

Proof. Divide through by n^3 and cancel all trivially zero terms:

$$\begin{aligned}\lim_{n\to\infty}\frac{3n^3+2n^2-n-1}{n^3+n+3} &= \lim_{n\to\infty}\frac{\frac{3n^3}{n^3}+\frac{2n^2}{n^3}-\frac{n}{n^3}-\frac{1}{n^3}}{\frac{n^3}{n^3}+\frac{n}{n^3}+\frac{3}{n^3}} \\ &= \lim_{n\to\infty}\frac{3+\frac{2}{n}-\frac{1}{n^2}-\frac{1}{n^3}}{1+\frac{1}{n^2}+\frac{3}{n^3}} \\ &= \lim_{n\to\infty}\frac{3+0-0-0}{1+0+0} \\ &= 3\end{aligned}$$

□

(e) $\lim\limits_{n\to\infty} \dfrac{n^2-2n-6}{n+1}$

Proof. Let $a_n = \frac{n^2-2n-6}{n+1}$. Perform polynomial division to find $a_n = n - 3 - \frac{3}{1+n}$. Notice that $\frac{3}{1+n} \leq 3$, so $a_n \geq n-6$. $n-6$ obviously diverges, so a_n diverges to ∞. □

Q04. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \dfrac{7+a_n}{6}$ for $n \geq 1$.

(a) By induction, show that $\{a_n\}$ is an increasing sequence that is bounded above by 2.

Proof. Let $n = 1$. $a_n = 1$ and $a_{n+1} = \frac{7+1}{6} = \frac{4}{3}$. Therefore, $a_1 < a_2 < 2$.

Suppose that $a_n < a_{n+1} < 2$ for some n. Then,

$$\begin{aligned} a_n &< a_{n+1} &&< 2 \\ 7 + a_n &< 7 + a_{n+1} &&< 9 \\ \frac{7+a_n}{6} &< \frac{7+a_{n+1}}{6} &&< \frac{3}{2} \\ a_{n+1} &< a_{n+2} &&< 2 \end{aligned}$$

By induction, $a_n < a_{n+1} < 2$ for all n. Therefore, $\{a_n\}$ is increasing and bounded above by 2. □

(b) Prove that this sequence is convergent and find $\lim_{n\to\infty} a_n$.

Proof. By the monotone convergence theorem, because $\{a_n\}$ is non-decreasing and bounded above, it must converge. We can therefore let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{7+a_n}{6} \\ &= \frac{7 + \lim_{n\to\infty} a_n}{6} \\ &= \frac{7+L}{6} \\ 6L &= 7 + L \\ L &= \frac{7}{5} \end{aligned}$$

□

Q05. Define a sequence $\{a_n\}$ by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$ for $n \geq 1$.

(a) By induction, show that $\{a_n\}$ is an increasing sequence that is bounded above by 3.

Proof. Let $n = 1$, so $a_n = \sqrt{2}$ and $a_{n+1} = \sqrt{2+\sqrt{2}} \approx 1.85$. As required, $a_n < a_{n+1} < 3$.

Suppose $a_n < a_{n+1} < 3$ for some n. Then,

$$\begin{aligned} a_n &< a_{n+1} &&< 3 \\ 2 + a_n &< 2 + a_{n+1} &&< 5 \\ \sqrt{2+a_n} &< \sqrt{2+a_{n+1}} &&< \sqrt{5} \end{aligned}$$

By induction, $a_n < a_{n+1} < 3$ for all n. Therefore, $\{a_n\}$ is increasing and bounded above by 3. □

(b) Prove that this sequence is convergent and find $\lim_{n\to\infty} a_n$.

Proof. By the monotone convergence theorem, because $\{a_n\}$ is bounded above and non-decreasing, it converges to a limit L. Recall that if $a_n \to L$, then $a_{n+1} \to L$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt{2 + a_n} \\
&= \sqrt{2 + \lim_{n\to\infty} a_n} \\
&= \sqrt{2 + L} \\
L^2 &= 2 + L \\
0 &= L^2 - L - 2 \\
0 &= (L-2)(L+1)
\end{aligned}$$

So L is either -1 or 2. Notice the recursive definition of a_n uses a square root, so $a_n \geq 0$. Therefore, L must be 2. $\square$