**MATH 137 Fall 2020: Practice Assignment 4**

**Q01.** Use the $\epsilon$–$\delta$ definition of limits to establish the following:

(a) $\lim_{x \to 3} 2x + 1 = 7$

*Proof.* First, recall the $\epsilon$–$\delta$ definition of a limit. We must show that for every $\epsilon > 0$, there is a $\delta$ so $|x - 3| < \delta$ implies $|2x + 1 - 7| = |2x - 6| < \epsilon$.

Let $\epsilon > 0$. Suppose $\delta = \frac{\epsilon}{2}$. If $0 < |x - 3| < \delta$, then $|x - 3| < \frac{\epsilon}{2}$. Now,

$$\begin{aligned} |x-3| &< \frac{\epsilon}{2} \\ 2|x-3| &< \epsilon \\ |2x-6| &< \epsilon \end{aligned}$$

as desired. Therefore, $\lim_{x \to 3} 2x + 1 = 7$. □

(b) $\lim_{x \to -1} 1 - 9x = 10$

*Proof.* Let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{9}$, and suppose $0 < |x - (-1)| = |x + 1| < \delta$. Now,

$$\begin{aligned} |x+1| &< \frac{\epsilon}{9} \\ 9|x+1| &< \epsilon \\ |9x+9| &< \epsilon \\ |-9x-9| &< \epsilon \\ |(1-9x)-(10)| &< \epsilon \end{aligned}$$

Therefore, by the $\epsilon$–$\delta$ definition of a limit, $\lim_{x \to -1} 1 - 9x = 10$. □

(c) $\lim_{x \to 5} 3 = 3$

*Proof.* First, notice that $|3 - 3|$ is always 0. Because we define $\epsilon > 0$, then $|3 - 3| < \epsilon$ for any choice of $\epsilon$. Therefore, we can arbitrarily let $\delta = 69$ since the choice of $\delta$ has no bearing on whether $|3 - 3| < \epsilon$, and say that by the $\epsilon$–$\delta$ definition, $\lim_{x \to 5} 3 = 3$. □

(d) $\lim_{x \to 2} x^2 - 4x + 4 = 0$

*Proof.* Let $\epsilon > 0$. Select $\delta = \sqrt{\epsilon}$. Suppose $0 < |x - 2| < \delta$, so

$$\begin{aligned} |x-2| &< \sqrt{\epsilon} \\ |(x-2)^2| &< \epsilon \\ |x^2-4x+4| &< \epsilon \\ |(x^2-4x+4)-0| &< \epsilon \end{aligned}$$

which is the $\epsilon$–$\delta$ definition of $\lim_{x \to 2} x^2 - 4x + 4 = 0$. □

(e) $\lim_{x \to 3} \frac{1}{x^2} = \frac{1}{9}$

*Proof.* Let $\epsilon > 0$. We can restrict $\delta \leq 1$ by selecting $\delta = \min(1, \frac{\epsilon}{7})$. Then, when $0 < |x-3| < \delta$, we have $2 < x < 4$, which means $|x+3| < 7$. It follows that

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{9}\right| &= \left|\frac{(x-3)(x+3)}{9x^2}\right| \\ &< |(x-3)(x+3)| \\ &= |x-3| \cdot |x+3| \\ &< 7|x-3| \\ &< 7\delta \\ &= \epsilon\end{aligned}$$

as required by the $\epsilon$–$\delta$ definition of $\lim_{x \to 3} \frac{1}{x^2} = \frac{1}{9}$ □

**Q02.** Let $f(x) > 0$ for all $x \neq a$ and assume $\lim_{x \to a} f(x) = L$ with $L \neq \pm\infty$. Use the definition of limits to show that $L \geq 0$. Hint: build a contradiction assuming $L < 0$.

*Proof.* Let $f(x)$ be a function that converges to $L$ at $x = a$, and is positive for all $x$. Then, by the $\epsilon$–$\delta$ definition of a limit, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. Suppose for a contradiction that $L < 0$.

Select $\epsilon = -L$ (since $L$ is negative). Then, we must be able to find a $\delta$ so $0 < |x-a| < \delta$ implies $|f(x) - L| < -L$. This is equivalently stated $L < f(x) - L < -L$ or $2L < f(x) < 0$. But $f(x) > 0$ for all $x$, so this inequality can never hold.

Therefore, $L < 0$ cannot be the limit, so $\lim_{x \to a} f(x) \geq 0$. □

**Q03.** For the following limits, $\lim_{x \to a} f(x)$, find a sequence $x_n$ such that $x_n \to a$, $x_n \neq a$ and then use this sequence to show that the limits do not exist.

(a) $\lim_{x \to 3} \frac{1}{x-3}$

*Proof.* Let $f(x) = \frac{1}{x-3}$. We will prove that the limit as $x$ approaches 3 does not exist.

Recall the Sequential Characterization of Limits. It provides that, given a function $f$ defined around $x = a$, the statement "$\lim_{x \to a} f(x)$ exists and is $L$" is equivalent to "for any sequence $\{x_n\}$ with $x_n \neq a$ and $x_n \to a$, $\lim_{n \to \infty} f(x_n) = L$."

Therefore, to show that the limit does not exist, it suffices to show the negation: there exists a sequence $\{x_n\}$ with $x_n \neq 3$ and $x_n \to 3$ where $f(x_n)$ diverges.

Let $x_n = 3 - \frac{1}{n}$. This sequence clearly converges to but is never equal to 3. Then,

$$f(x_n) = \frac{1}{3 - \frac{1}{n} - 3} = -n$$

$f(x_n)$ clearly diverges to negative infinity, so the limit does not exist. □

(b) $\lim_{x \to 0} \ln|x|$

*Proof.* Let $f(x) = \ln|x|$. As above, we must prove the limit at $x = 0$ does not exist.

Again, using the Sequential Characterization of Limits, let $x_n = \frac{1}{x}$, which converges to 0 but is never equal to 0. Then, when $n \geq 1$,

$$f(x_n) = \ln\left|\frac{1}{n}\right| = -\ln n,$$

a sequence which diverges to $-\infty$.

Therefore, the limit does not exist. □

**Q04.** Prove that $\lim\limits_{x\to 3} \dfrac{x^2-9}{|x-3|}$ does not exist by finding two sequences $x_n$ and $y_n$ that both converge to 3, $x_n \neq 3$, $y_n \neq 3$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$. Explain why this proves the limit does not exist.

*Proof.* Suppose for a contradiction that the limit exists and is equal to $L$. Then, according to the Sequential Characterization of Limits, we can find any sequence $\{a_n\}$ with $a_n \to 3$ but $a_n \neq 3$, and $f(a_n)$ will converge to $L$.

Consider the sequence $x_n = 3 + \frac{1}{x}$ for $n \geq 1$, which converges to and is never equal to 3. Substituting,

$$f(x_n) = \frac{\left(3+\frac{1}{n}\right)^2 - 9}{\left|3 + \frac{1}{n} - 3\right|} = \frac{9 + \frac{6}{n} + \frac{1}{n^2} - 9}{\frac{1}{n}} = 6 + \frac{1}{n}$$

which converges to 6. This implies that $L = 6$.

However, we could also consider the sequence $y_n = 3 - \frac{1}{x}$ for $n \geq 1$, which also satisfies the same constraints. We would instead have $f(y_n)$ as

$$f(y_n) = \frac{\left(3-\frac{1}{n}\right)^2 - 9}{\left|3 - \frac{1}{n} - 3\right|} = \frac{9 - \frac{6}{n} + \frac{1}{n^2} - 9}{\frac{1}{n}} = -6 + \frac{1}{n}$$

which converges to $-6$. This implies that $L = -6$.

Since the limit cannot have both values, it cannot exist. □

**Q05.** Compute the following limits using any method. If they do not exist, prove it.

(a) $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4}$

*Solution.* Notice that for all $x \neq 4$,

$$\frac{x^2-16}{x-4} = \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}} = x + 4$$

so $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} = \lim\limits_{x\to 4} x + 4 = 4 + 4 = 8$ □

(b) $\lim\limits_{x\to 2} \dfrac{x^3 - 6x^2 + 12x - 8}{x-2}$

*Solution.* Notice that for all $x \neq 2$,

$$\frac{x^3 - 6x^2 + 12x - 8}{x - 2} = \frac{(x-2)^3}{x-2} = (x-2)^2$$

so $\lim_{x \to 2} \frac{x^3 - 6x^2 + 12x - 8}{x - 2} = \lim_{x \to 2} (x-2)^2 = 0$ □

(c) $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$

*Solution.* Notice that for all $x \neq 16$,

$$\frac{\sqrt{x} - 4}{x - 16} = \frac{\sqrt{x} - 4}{(\sqrt{x} - 4)(\sqrt{x} + 4)} = \frac{1}{\sqrt{x} + 4}$$

so $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} = \lim_{x \to 16} \frac{1}{\sqrt{x} + 4} = \frac{1}{8}$ □

(d) $\lim_{x \to \pi/2} \frac{\tan^2 x + 1}{\sec^2 x}$

*Solution.* Recall that $\tan^2 \theta + 1 = \sec^2 \theta$ for all real $\theta$. Then,

$$\lim_{x \to \pi/2} \frac{\tan^2 x + 1}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\sec^2 x}{\sec^2 x} = \lim_{x \to \pi/2} 1 = 1$$ □

(e) $\lim_{x \to 3} \frac{|3x - 9|}{3 - x}$

*Solution.* Consider the limit from above. If $x > 3$ then $3x - 9 > 0$, so we can drop the absolute value:

$$\lim_{x \to 3^+} \frac{|3x - 9|}{3 - x} = \lim_{x \to 3^+} \frac{3x - 9}{3 - x} = \lim_{x \to 3^+} \frac{-3(3 - x)}{3 - x} = \lim_{x \to 3^+} -3 = -3$$

Consider the limit from below. If $x < 3$ then $3x - 9 < 0$, so $|3x - 9| = -(3x - 9)$:

$$\lim_{x \to 3^-} \frac{|3x - 9|}{3 - x} = \lim_{x \to 3^-} \frac{9 - 3x}{3 - x} = \lim_{x \to 3^-} \frac{3(3 - x)}{3 - x} = \lim_{x \to 3^-} 3 = 3$$

Since the two one-sided limits do not agree, the limit does not exist. □

**Q06.** Consider the function

$$f(x) = \begin{cases} 1 + \frac{1}{x} & x < b \\ 1 + x & x > b \end{cases}$$

Determine all values of $b \in \mathbb{R}$ for which $\lim_{x \to b} f(x)$ exists. Find the limit in each case. Prove that the limit does not exist for any other choice of $b$.

*Proof.* First, recall that the one-sided limits must exist and be equal for the two-sided limit to exist. Therefore, $\lim_{x\to b} f(x)$ exists only when both $\lim_{x\to b^-} f(x)$ and $\lim_{x\to b^+} f(x)$ exist.

Consider the limit from below. Then, we only need consider when $x < b$, which means $f(x) = 1 + \frac{1}{x}$. Now, consider cases for zero and non-zero $b$. When $b = 0$, we have $1 + \frac{1}{x}$ diverging to $-\infty$ as $x \to b$. For non-zero $b$, we can simply apply the arithmetic rules and get the limit from below as $1 + \frac{1}{b}$.

Consider the limit from above. Then, $x > b$, so we have $f(x) = 1 + x$. For all $b$, we can apply arithmetic rules to have the limit from above as $1 + b$.

Now, we can compare our results. For $b = 0$, the two-sided limit clearly does not exist, because the left side diverges and the right side converges. For $b \neq 0$, the limits are only equal when $1 + \frac{1}{b} = 1 + b$. Then, $b^2 = 1$, so $b$ can only be $-1$ or 1. At theses points, the limit is $1 + (-1) = 0$ and $1 + 1 = 2$, respectively. □

**Q07.** Compute the following limits without using l'Hôpital's rule:

(a) $\lim_{x\to 0} \dfrac{\sin^2 x}{x^2}$

*Solution.* Recall the FTL, $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$. Now, rearrange and apply:

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{x}\right) = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x} = 1 \cdot 1 = 1$$ □

(b) $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

*Proof.* First, algebraically manipulate the expression inside the limit to simplify:

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}$$

□

(c) $\lim_{x\to 2} \dfrac{\sin\big(3(x^2 - 4)\big)}{x - 2}$

*Solution.* To apply FTL, factor and multiply by $1 = \frac{3(x+2)}{3(x+2)}$.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sin\big(3(x^2 - 4)\big)}{x - 2} &= \lim_{x\to 2} \frac{\sin(3(x-2)(x+2))}{x - 2} \\
&= \lim_{x\to 2} 3(x+2) \frac{\sin(3(x-2)(x+2))}{3(x+2)(x-2)} \\
&= \lim_{x\to 2} 3(x+2) \\
&= 12
\end{aligned}$$ □

(d) $\lim_{x\to 0} \dfrac{\sin x^2}{\sqrt{|x^3|}}$

*Solution.* Again, to apply FTL, creatively multiply by $1 = \frac{x^{1/2}}{x^{1/2}}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x^2}{\sqrt{|x^3|}} &= \lim_{x\to 0} \frac{\sin x^2}{\sqrt{\sqrt{(x^3)^2}}} \\
&= \lim_{x\to 0} \frac{\sin x^2}{((x^3)^2)^{1/4}} \\
&= \lim_{x\to 0} \frac{\sin x^2}{x^{3/2}} \\
&= \lim_{x\to 0} x^{1/2} \frac{\sin x^2}{x^2} \\
&= \lim_{x\to 0} x^{1/2} \\
&= 0
\end{aligned}$$

□