MATH 137 Fall 2020: Practice Assignment 5

Q01. Compute the following limits using the fact that $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ and $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ for any $p > 0$.

(a) $\lim_{x\to\infty} \frac{\sqrt{x} + \ln x - x}{1 - \ln e^{2x}}$

Solution. Divide through by x:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x} + \ln x - x}{1 - \ln e^{2x}} &= \lim_{x\to\infty} \frac{\sqrt{x} + \ln x - x}{1 - 2x} \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{x}}{x} + \frac{\ln x}{x} - 1}{\frac{1}{x} - 2} \\
&= \frac{0 + 0 - 1}{0 - 2} \\
&= \frac{1}{2}
\end{aligned}$$

□

(b) $\lim_{x\to\infty} e^{-x}\left(1 - x\sqrt{e^x}\right)$

Solution. Distribute and simplify:

$$\begin{aligned}
\lim_{x\to\infty} e^{-x}\left(1 - x\sqrt{e^x}\right) &= \lim_{x\to\infty} \left(e^{-x} - e^{-x}x\sqrt{e^x}\right) \\
&= \lim_{x\to\infty} \left(0 - e^{-x}xe^{x/2}\right) \\
&= \lim_{x\to\infty} \frac{x}{e^{x/2}} \\
&= 0
\end{aligned}$$

□

(c) $\lim_{x\to\infty} \frac{\frac{\ln x}{x^p}}{\frac{x^p}{e^x}}$ for $p > 0$.

Solution. Recall that a product of divergences diverges.

$$\lim_{x\to\infty} \frac{\frac{\ln x}{x^p}}{\frac{x^p}{e^x}} = \lim_{x\to\infty} e^x \ln x = \infty$$

□

(d) $\lim_{x\to\infty} \frac{(\ln x)^e}{x}$

Solution. Rewrite as $\left(\frac{\ln x}{x^{1/e}}\right)^e$. This follows the given pattern, so the limit is $0^e = 0$. □

Q02. Find all asymptotes (both horizontal and vertical) of $f(x) = \frac{1}{-2x^2 + 2}$.

Solution. Notice that the limit as $x \to \infty$ is 0, so $y = 0$ is an asymptote. Vertical asymptotes of a rational function occur only when the numerator is 0 but the denominator is non-zero. The denominator factors to $-2(x-1)(x+1)$, so $x = \pm 1$ are asymptotes. □

Q03. Prove that the function $f(x) = 2x^2+9$ is continuous at $x = 2$ using the ϵ–δ definition of continuity.

Proof. We must show that $\lim_{x\to a} f(x) = f(a)$ for $a = 2$. Specifically, $\lim_{x\to 2}(2x^2 + 9) = 17$.

This means that for any $\epsilon > 0$, we can find a δ where $0 < |x - 2| < \delta$ implies $|2x^2 - 8| < \epsilon$.

Let $\epsilon > 0$. Choose $\delta = \min(\{\frac{\epsilon}{8}, 2\})$, limiting δ to be at most 2. Also, suppose that $0 < |x - 2| < \delta$. Then, we have $|x + 2| < 4$ and $|x - 2| < \frac{\epsilon}{8}$. Multiplying:

$$\begin{aligned}
|x-2||x+2| &< \frac{\epsilon}{8} \cdot 4 \\
\left|x^2 - 4\right| &< \frac{\epsilon}{2} \\
\left|2x^2 - 8\right| &< \epsilon
\end{aligned}$$

which is exactly what we needed to show.

Therefore, by the ϵ–δ definition of continuity, $f(x)$ is continuous at $x = 2$. □

Q04. Let f be a function defined as

$$f(x) = \begin{cases} \dfrac{x^2-4}{x^2+x-6}\cos x^2 & x \neq -3, 2 \\ 0 & x = -3, 2 \end{cases}$$

Find the intervals where f is continuous. Justify your answer.

Proof. First, simplify f by factoring:

$$\begin{aligned}
f(x) &= \begin{cases} \dfrac{(x-2)(x+2)}{(x-2)(x+3)}\cos x^2 & x \neq -3, 2 \\ 0 & x = -3, 2 \end{cases} \\
&= \begin{cases} \dfrac{x+2}{x+3}\cos x^2 & x \neq -3, 2 \\ 0 & x = -3, 2 \end{cases}
\end{aligned}$$

Note that cancelling the $x - 2$ factors is allowed since the term is only defined when $x \neq 2$.

By the arithmetic rules for continuity, f is continuous everywhere except possibly at $x = -3$ and $x = 2$.

Consider these two points.

At $x = -3$, we define $f(-3) = 0$. However, the limit from above blows up to negative infinity (and from below to positive infinity). Since the limit does not exist, the function is not continuous.

At $x = 2$, we again define $f(2) = 0$. Applying arithmetic limit rules, the limit is $\frac{2+2}{2+3}\cos 2^2 = \frac{4}{5}\cos 4$. This does not equal the value of the function, 0, so the function is not continuous.

Therefore, f is continuous on $\mathbb{R} \setminus \{-3, 2\}$. □

Q05. Let

$$f(x) = \begin{cases} cx^2 + 2x & x > 2 \\ x^3 - cx & x \leq 2 \end{cases}$$

Find the value c such that $f(x)$ is continuous on $\mathbb{R}$. Justify your answer.

Proof. By the arithmetic rules for continuity, $f(x)$ is clearly continuous on $\mathbb{R} \setminus \{2\}$.

For $f(x)$ to be continuous at $x = 2$, the one-sided limits must agree and equal $f(2)$. Since $f(2)$ is defined using the definition for $x > 2$, we need only compare the two cases.

By the limit rules for polynomials, we have:

$$\begin{aligned} \lim_{x \to 2^+} (cx^2 + 2x) &= \lim_{x \to 2^-} (x^3 - cx) \\ c(2)^2 + 2(2) &= (2)^3 - c(2) \\ 4c + 4 &= 8 - 2c \\ c &= \frac{2}{3} \end{aligned}$$

□

Q06. Show that if a function is continuous at $x = 0$ and satisfies the following, it is it is continuous everywhere.

Hint: You may use the fact that $\lim_{x \to a} f(x) = f(a)$ is equivalent to $\lim_{h \to 0} f(a + h) = f(a)$.

(a) $f(x + y) = f(x) + f(y)$

Proof. Let f be a function continuous at $x = 0$. By definition, $\lim_{x \to 0} f(x) = f(0)$. We must show that for all a, $\lim_{x \to a} f(x) = f(a)$.

Let a be an arbitrary value in the domain of f. Recall that $f(a+h) = f(a) + f(h)$, so

$$\lim_{h \to 0} f(a + h) = \lim_{h \to 0} f(a) + \lim_{h \to 0} f(h) = f(a) + f(0) = f(a + 0) = f(a)$$

By the above hint, this is equivalent to saying $\lim_{x \to a} f(x) = f(a)$. □

(b) $f(x + y) = f(x)f(y)$

Proof. Let f be a function continuous at $x = 0$, i.e., $\lim_{x \to 0} f(x) = f(0)$. Again, we must show that for all a, $\lim_{x \to a} f(x) = f(a)$.

Let a be an arbitrary value in the domain of f. Since $f(x + y) = f(x)f(y)$:

$$\lim_{h \to 0} f(a + h) = \lim_{h \to 0} f(a) \cdot \lim_{h \to 0} f(h) = f(a) \cdot f(0) = f(a + 0) = f(a)$$

By the above hint, this is equivalent to saying $\lim_{x \to a} f(x) = f(a)$. □