MATH 137 Fall 2020: Practice Assignment 6

Q01. For $f(x) = \dfrac{x+1}{x-1}$, find $f'(x)$ using the limit definition.

Solution. Apply the Newton quotient:

$$\begin{aligned}
f'(x) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{x+1}{x-1} - \frac{a+1}{a-1}}{x-a} \\
&= \lim_{x\to a} \frac{(x+1)(a-1)-(a+1)(x-1)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{(xa+a-x-1)-(xa-a+x-1)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{-2(x-a)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{-2}{(x-1)(a-1)} \\
&= -\frac{2}{(x-1)^2}
\end{aligned}$$

□

Q02. Let $f(x) = \dfrac{ax+b}{ax-b}$ where $a \neq 0$, $b \neq 0$.

(a) Find $f'(x)$ using any method.

Solution. First, notice that $f(x)$ is undefined at $x = \frac{b}{a}$, so we differentiate along all $x \neq \frac{b}{a}$. Apply the quotient and linear function rules:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax+b}{ax-b}\right) &= \frac{(ax-b)\frac{\mathrm{d}}{\mathrm{d}x}(ax+b)-(ax+b)\frac{\mathrm{d}}{\mathrm{d}x}(ax-b)}{(ax-b)^2} \\
&= \frac{(ax-b)a-(ax+b)a}{(ax-b)^2} \\
&= \frac{a(-2b)}{(ax-b)^2} \\
&= -\frac{2ab}{(ax-b)^2}
\end{aligned}$$

□

(b) Show that for $x \neq \frac{b}{a}$, $abf'(x) < 0$.

Proof. Let a and b be non-zero reals, and let $x \neq \frac{b}{a}$. Then,

$$\begin{aligned}
abf'(x) &= ab\left(\frac{2ab}{(ax-b)^2}\right) \\
&= -\frac{2a^2b^2}{(ax-b)^2}
\end{aligned}$$

Recall that the square of any non-zero number is positive. Then, we have that $a^2 > 0$, $b^2 > 0$, and $(ax-b)^2 > 0$. The last one also implies $\frac{1}{(ax-b)^2} > 0$. Multiplying,

$$\begin{aligned} \frac{a^2b^2}{(ax-b)} &> 0 \\ -2\frac{a^2b^2}{(ax-b)} &< 0 \\ abf'(x) &< 0 \end{aligned}$$

□

Q03. In each case, find $f'(x)$ using any method.

(a) $f(x) = 5^x \sin x + (x^3 + x^2)\cos x$.

Solution. Apply arithmetic rules and recall that $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(5^x \sin x) + \frac{\mathrm{d}}{\mathrm{d}x}\big((x^3+x^2)\cos x\big) \\ &= (5^x \frac{\mathrm{d}}{\mathrm{d}x}\cos x + \cos x \frac{\mathrm{d}}{\mathrm{d}x} 5^x) + (\cos x \frac{\mathrm{d}}{\mathrm{d}x}(x^3+x^2) + (x^3+x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cos x) \\ &= 5^x \sin x + \ln 5 \cos x 5^x + \cos x(3x^2+2x) + (x^3+x^2)\sin x \\ &= \sin x(x^3 + x^2 + 5^x) + \cos x(5^x \ln 5 + 3x^2 + 2x) \end{aligned}$$

□

(b) $f(x) = \dfrac{x^2+x-2}{x^3+6}$.

Solution. Apply the quotient rule, excepting $x = \sqrt[3]{-6}$ from the domain:

$$\begin{aligned} f'(x) &= \frac{(x^3+6)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+x-2) - (x^2+x-2)\frac{\mathrm{d}}{\mathrm{d}x}(x^3+6)}{(x^3+6)^2} \\ &= \frac{(x^3+6)(2x+1) - (x^2+x-2)3x^2}{(x^3+6)^2} \\ &= -\frac{x^4+2x^3-6x^2-12x-6}{(x^3+6)^2} \end{aligned}$$

□

(c) $f(x) = \sqrt{2\tan^2 x + 3}$.

Solution. Apply the chain rule, recalling that $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}\sqrt{2\tan^2 x+3}}{\mathrm{d}(2\tan^2 x + 3)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\big(2\tan^2 x + 3\big) \\ &= \frac{1}{2\sqrt{2\tan^2 x+3}} \cdot 2\frac{\mathrm{d}\tan^2 x}{\mathrm{d}\tan x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\tan x \\ &= \frac{1}{2\sqrt{2\tan^2 x+3}} \cdot \left(2\frac{\mathrm{d}\tan^2 x}{\mathrm{d}\tan x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\tan x\right) \\ &= \frac{1}{2\sqrt{2\tan^2 x+3}} \cdot 4\tan x \sec^2 x \\ &= \frac{2\tan x \sec^2 x}{\sqrt{2\tan^2 x + 3}} \end{aligned}$$

□

(d) $f(x) = 2^{\sin(\sec x)}$.

Solution. Again, simply apply the chain rule repeatedly.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}(2^{\sin(\sec x)})}{\mathrm{d}(\sin(\sec x))} \cdot \frac{\mathrm{d}\sin(\sec x)}{\mathrm{d}(\sec x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sec x \\ &= 2^{\sin(\sec x)} \ln(2)\cos(\sec x)\sec(x)\tan(x) \end{aligned}$$

□

Q04. In each case, determine the equation of the tangent to $y = f(x)$ at the point where $x = a$.

(a) $f(x) = x^2$, $a = 3$.

Solution. By the power rule, $f'(x) = 2x$. Recall the formula for the equation of a tangent: $L_a^f(x) = f(a) + f'(a)(x - a)$. Apply it:

$$\begin{aligned} L_3^f(x) &= f(3) + f'(3)(x-3) \\ &= 3^2 + 2(3)(x-3) \\ y &= 6x - 9 \end{aligned}$$

□

(b) $f(x) = \cos x$, $a = -\dfrac{3\pi}{4}$.

Solution. Again, apply the linear approximation formula, knowing $f'(x) = -\sin x$.

$$\begin{aligned} L_{-3\pi/4}^f &= f(-\,3\pi/4) + f'(-\,3\pi/4)\left(x - \frac{3\pi}{4}\right) \\ &= \cos(-\,3\pi/4) - \sin(-\,3\pi/4)\left(x - \frac{3\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) \\ y &= \frac{\sqrt{2}}{2}x + \frac{3\pi - 2\sqrt{2}}{4} \end{aligned}$$

□

(c) $f(x) = e^x$, $a = \ln \pi$.

Solution. Nothing new or fancy here. Even less so since $f'(x) = e^x$.

$$\begin{aligned} L_{\ln\pi}^f &= f(\ln\pi) + f'(\ln\pi)(x - \ln\pi) \\ &= e^{\ln\pi} + e^{\ln\pi}(x - \ln\pi) \\ &= \pi + \pi(x - \ln\pi) \\ y &= \pi x - \ln\pi - \pi \end{aligned}$$

□

(d) $f(x) = 4^x$, $a = -3$.

Solution. Recall the derivative of an exponential: $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$.

$$\begin{aligned} L_{-3}^f &= f(-3) + f'(-3)(x+3) \\ &= 4^{-3} + 4^{-3}\ln 4(x+3) \\ &= \frac{\ln 2}{32}x + \frac{3\ln 4 + 1}{64} \end{aligned}$$

□

Q05. Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in each case.

(a) $y = \cos x^2$.

Solution. Apply the chain rule to find the first derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\cos\left(x^2\right)}{\mathrm{d}(x^2)} \cdot \frac{\mathrm{d}x^2}{\mathrm{d}x} = -2x\sin x^2.$$

Apply the product and chain rule to find the second derivative:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-2x\sin x^2\right) \\ &= -2\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sin x^2\right) \\ &= -2\left(x\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 + \sin x^2\frac{\mathrm{d}}{\mathrm{d}x}x\right) \\ &= -2\left(x(2x\cos x^2) + \sin x^2(1)\right) \\ &= -4x^2\cos x^2 - 2\sin x^2.\end{aligned}$$

□

(b) $y = \cos^2 x$.

Solution. Follow the same procedure as above, remembering that $\cos^2 x = (\cos x)^2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}\cos^2 x}{\mathrm{d}\cos x} \cdot \frac{\mathrm{d}\cos x}{\mathrm{d}x} = -2\cos x\sin x.$$

We can simplify this to $-\sin 2x$ using the double angle identity. Then,

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(-\sin 2x) \\ &= -\frac{\mathrm{d}\sin(2x)}{\mathrm{d}(2x)} \cdot \frac{\mathrm{d}(2x)}{\mathrm{d}x} \\ &= -2\cos 2x.\end{aligned}$$

□

Q06.

(a) Use the Chain Rule to prove that the derivative of an even function is odd.

Proof. Recall that an even function f is one where $f(-x) = f(x)$ for all x, and an odd function g is one where $g(-x) = -g(x)$ for all x. Let f be even. We must show that $f'(-x) = -f'(x)$.

Notice that $(f(-x))' = f'(-x) \cdot (-x)' = -f'(-x)$. However, because f is even, this is equal to $(f(x))' = f'(x)$. That is, $-f'(-x) = f'(x)$ and it follows that f' is odd. □

(b) Using ONLY the Chain Rule and the Product Rule (and not the Reciprocal/Quotient rules), give an alternative proof of the Quotient Rule. [Hint: $\frac{f(x)}{g(x)} = f(x)(g(x))^{-1}$].

Proof. Let f and g be differentiable functions, and let $h(x) = \frac{f(x)}{g(x)}$. According to the above hint, write h as $f(x) \cdot (g(x))^{-1}$.

Now, apply the product rule:

$$h'(x) = f(x)(g(x)^{-1})' + f'(x)(g(x))^{-1}$$

We can evaluate the derivative of $g(x)^{-1}$ using the power and chain rules:

$$(g(x)^{-1})' = (-1)g(x)^{-2} \cdot g'(x) = -\frac{g'(x)}{g(x)^2}$$

Substiting back in and simplifying, we arrive at the quotient rule:

$$-\frac{f(x)g'(x)}{g(x)^2} + \frac{f'(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$ □

Q07. If $y = f(u)$ and $u = g(x)$ where f and g are twice differentiable functions, prove that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}.$$

Proof. Let y be dependent on u and u be dependent on x. Then, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

And, differentiating both sides, we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

which is just a product, so we may apply the product rule. This gives

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right) \cdot \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \\ &= \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\end{aligned}$$

exactly as desired. □