MATH 137 Fall 2020: Practice Assignment 7

Q01. Let $f(x) = \sin x$.

(a) Determine the equation for the linear approximation to $f(x)$ at $x = a$, L_a^f, for any $a \in \mathbb{R}$.

Solution. Apply the formula, knowing $(\sin x)' = \cos x$:

$$\begin{aligned} L_a^f &= f(a) + f'(a)(x-a) \\ &= x\cos a + \sin a - a\cos a \end{aligned}$$

□

(b) Determine the equation for the linear approximation to $f(x)$ at $x = \frac{\pi}{3}$, $L_{\pi/3}^f$.

Solution. Substituting for $x = \frac{\pi}{3}$ into part (a):

$$\begin{aligned} L_{\pi/3}^f &= x\cos\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\pi}{3}\cos\frac{\pi}{3} \\ &= \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6} \\ &= \frac{1}{2}x + \frac{3\sqrt{3}-\pi}{6} \end{aligned}$$

□

(c) Use your answer to (b) to approximate $\sin(1)$.

Solution. Substituting for $x = 1$ into part (b),

$$L_{\pi/3}^f = \frac{1}{2}(1) + \frac{3\sqrt{3}-\pi}{6} = \frac{3+3\sqrt{3}-\pi}{6} \approx 0.842.$$

□

(d) Since $f''(x) = -\sin x$, we can safely say that $f''(x) \leq 1$. Given this fact, what is the maximum error (worst-case scenario) for your answer in (c)?

Solution. Recall that if $|f''(x)| \leq M$ for all x in an interval I containing a, then the worst case error is given by $|f(x) - L_a(x)| \leq \frac{M}{2}(x-a)^2$.
Substitute $x = 1$, $M = 1$, and $a = \frac{\pi}{3}$:

$$\begin{aligned} |f(1) - L_{\pi/3}| &\leq \frac{1}{2}\left(1 - \frac{\pi}{3}\right)^2 \\ &= \frac{1}{2}\left(\frac{2\pi}{3}\right)^2 \\ &= \frac{2\pi^2}{9} \end{aligned}$$

Therefore, the maximum error is $\frac{2\pi}{9}$.

□

Q02. Rich bought a yoga ball that is made of material which, when the ball is properly inflated, is a sphere with outer radius R. The manufacturer has determined that the material can tolerate a 4% "stretch" beyond specifications, meaning that if the ball is inflated in such a way that the surface area increases by more than 4% of the actual size, the material will rupture and the ball will deflate rather suddenly.

In the instructions for inflation, consumers are told to inflate the ball so that one side touches a wall and the other side touches a box placed $2R$ units away from the wall. Rich uses a ruler to measure $2R$ units from a wall, and then inflates the ball according to the instructions. If Rich's ruler and Rich's measurement skills create an error of 3% in excess of what he thinks he is measuring (that is, instead of inflating to a diameter of $2R$ it is inflated to a diameter of $1.03(2R)$), should we expect the ball to survive this initial inflation?

Use the technique of estimating change to investigate.

Solution. The formula for a surface area of a sphere is $A(r) = 4\pi r^2$, so we also have $A'(r) = 8\pi r$. Recall that we may estimate change by saying $\Delta A \cong A'(r_0)\Delta r$. We let $r_0 = R$ and $\Delta r = 0.03R$. Then, $\Delta A = 0.24\pi R^2$.

Finally, $\dfrac{\Delta A}{A(R)} = \dfrac{0.24\pi R^2}{4\pi R^2} = 0.06$, which is a 6% change.

Rich should duck and prepare for a rather sudden deflation. □

Q03.

(a) For $a \in \mathbb{R}$, $a > 0$, write down the recursive sequence that Newton's Method would generate for $f(x) = x^2 - a$. That is, write the formula for x_{n+1} in terms of x_n.

Solution. The recursive sequence is:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= \frac{2x_n^2 - x_n^2 + a}{2x_n} \\ &= \frac{x_n^2 + a}{2x_n} \end{aligned}$$

□

(b) Use Newton's Method to approximate each of the following, correct to 4 decimal places.

(i) $\sqrt{7}$ (use initial guess $x_1 = 3$)

Solution. Recursively apply the above formula with $a = 7$:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= \frac{(3)^2 + 7}{2(3)} &&= \frac{8}{3} &&\cong 2.6667 \\ x_3 &= \frac{(8/3)^2 + 7}{2(8/3)} &&= \frac{127}{48} &&\cong 2.6458 \\ x_3 &= \frac{(127/48)^2 + 7}{2(127/48)} &&= \frac{32257}{12192} &&\cong 2.6458 \end{aligned}$$

Since the figures agree to 4 decimal places, $\sqrt{7} \cong 2.6458$. □

(ii) $\sqrt{\pi}$ (use initial guess $x_1 = 2$)

Solution. Recursively apply the above formula with $a = \pi$:

$$\begin{aligned} x_1 &= 2 \\ x_2 &= \frac{(2)^2+\pi}{2(2)} && = 1 + \frac{\pi}{4} && \cong 1.7854 \\ x_3 &= \frac{(1+\frac{\pi}{4})^2+\pi}{2(1+\frac{\pi}{4})} && = \frac{\pi^2+24\pi+16}{8\pi+32} && \cong 1.7725 \\ x_4 &= \frac{(\frac{\pi^2+24\pi+16}{8\pi+32})^2+\pi}{2(\frac{\pi^2+24\pi+16}{8\pi+32})} && = \text{a fat mess} && \cong 1.7725 \end{aligned}$$

Since the figures agree to 4 decimal places, $\sqrt{\pi} \cong 1.7725$. □

Q04. Consider the function $f(x) = \dfrac{6x+1}{3x+5}$.

(a) What is the domain of f?

Solution. Apply rules for rational functions: $x \in \mathbb{R} \setminus \{-\frac{5}{3}\}$. □

(b) Find $f'(x)$.

Solution. Apply the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(3x+5)(6x+1)' - (6x+1)(3x+5)'}{(3x+5)^2} \\ &= \frac{6(3x+5) - 3(6x+1)}{(3x+5)^2} \\ &= \frac{27}{(3x+5)^2} \end{aligned}$$

□

(c) There is only one point $c \in \mathbb{R}$ where $f(c) = 0$, find it directly.

Solution. Notice that the numerator is zero when $x = -\frac{1}{6}$, and the denominator is not. Therefore, $f(-\frac{1}{6}) = 0$. □

(d) Now, starting with $x_1 = 5$ (a particularly foolish choice), perform 3 iterations of Newton's Method. Use 5 decimal places.

Solution. Recall the formula for the Newton's Method sequence: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. We can substitute $f(x) = \frac{6x+1}{3x+5}$ and $f'(x) = \frac{27}{(3x+5)^2}$ and simplify:

$$\begin{aligned} x_{n+1} &= x_n - \frac{\frac{6x_n+1}{3x_n+5}}{\frac{27}{(3x_n+5)^2}} \\ &= x_n - \frac{(6x_n+1)(3x_n+5)}{27} \\ &= -\frac{1}{27}(18x_n^2 + 60x_n + 5) \end{aligned}$$

Now, beginning with $x_1 = 5$, apply Newton's Method:

$$
\begin{aligned}
x_1 &= 5 \\
x_2 &= -\frac{1}{27}(18(5)^2 + 60(5) + 5) &&= -\frac{755}{27} &&\cong -27.96296 \\
x_3 &= -\frac{1}{27}(18(-755/27)^2 + 60(-755/27) + 5) &&= -\frac{1004555}{2187} &&\cong -459.33013 \\
x_3 &= -\frac{1}{27}(18(-\frac{1004555}{2187})^2 + 60(-\frac{1004555}{2187}) + 5) &&= -\frac{2003617741355}{14348907} &&\cong -139635.56537
\end{aligned}
$$

This is. . . not converging. □

(e) It is clear that starting with $x_1 = 5$ will not lead us to the root. In fact, the sequence generated by Newton's Method in this case diverges. Prove that using $x_1 = 5$ as a starting value, Newton's Method will not converge to the root of this function. (Hint: show the recursive sequence you get from Newton's Method is strictly decreasing).

Proof. We that the sequence $\{x_n\}$ is strictly decreasing along the relevant domain: Specifically, that $x_{n+1} < x_n$, that is,

$$
\begin{aligned}
-\frac{1}{27}(18x_n^3 + 60x_n + 5) &< x_n \\
18x_n^2 + 60x_n + 5 &> 27x_n \\
18x_n^2 + 33x_n + 5 &> 0 \\
(3x_n + 5)(6x_n + 1) &> 0
\end{aligned}
$$

This shows that $\{x_n\}$ is decreasing whenever $x_n < -\frac{5}{3}$ or $x_n > -\frac{1}{6}$. Once x_n falls into the former interval, x_{n+1} will be lower than x_n, which is still in the interval, so x_{n+2} is lower than x_{n+1}, *ad infinitum.*

Since $x_2 = -\frac{755}{27} < -\frac{5}{3}$, the sequence is infinitely decreasing. □

Q05. Using the fact that $(\ln x)' = \dfrac{1}{x}$, prove that $(\log_a x)' = \dfrac{1}{x \ln a}$ for $a > 0$, $a \neq 1$.

Proof. Let $0 < a \neq 1$. Recall the change of base formula: $\log_a x = \frac{\ln x}{\ln a}$.

Since $\frac{1}{\ln a}$ is a constant with respect to x, we may pull it out of the derivative, and it immediately follows that $(\log_a x)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x \ln a}$. □

Q06. Find $f'(x)$ if

(a) $f(x) = \sqrt{\log_{10}(7 + \sin x)}$

Solution. Apply the chain rule a bunch of times:

$$
\begin{aligned}
f'(x) &= (\log_{10}(7 + \sin x)^{1/2})' \\
&= \frac{1}{2}(\log_{10}(7 + \sin x)^{-1/2})(\log_{10}(7 + \sin x))' \\
&= \frac{1}{2\sqrt{\log_{10}(7 + \sin x)}} \cdot \frac{1}{(7 + \sin x)\ln 10} \cdot (7 + \sin x)' \\
&= \frac{\cos x}{2\sqrt{\log_{10}(7 + \sin x)}(7 + \sin x)\ln 10}
\end{aligned}
$$

□

(b) $f(x) = \arcsin\big(\tan x + x^3\big)e^x$

Solution. Recall that $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, and apply the product and chain rule:

$$\begin{aligned} f'(x) &= (\arcsin\big(\tan x + x^3\big))'e^x + \arcsin\big(\tan x + x^3\big)(e^x)' \\ &= \frac{(\tan x + x^3)'}{1-(\tan x + x^3)^2}e^x + \arcsin\big(\tan x + x^3\big)e^x \\ &= \frac{\sec^2 x + 3x^2}{1-(\tan x + x^3)^2}e^x + \arcsin\big(\tan x + x^3\big)e^x \end{aligned}$$

□

(c) $f(x) = \frac{x}{\arctan(x^2+1)}$

Solution. Recall that $(\arctan x)' = \frac{1}{1+x^2}$, and apply the quotient and chain rule:

$$\begin{aligned} f'(x) &= \frac{\arctan\big(x^2+1\big)(x)' - (x)(\arctan\big(x^2+1\big))'}{\arctan^2(x^2+1)} \\ &= \frac{\arctan\big(x^2+1\big) - x\frac{(x^2+1)'}{1+(x^2+1)^2}}{\arctan^2(x^2+1)} \\ &= \frac{1}{\arctan(x^2+1)} - \frac{\frac{x(2x)}{x^4+2x^2+2}}{\arctan^2(x^2+1)} \\ &= \frac{1}{\arctan(x^2+1)} - \frac{2x^2}{(x^4+2x^2+2)\arctan^2(x^2+1)} \end{aligned}$$

□

Q07. Let $f(x) = \frac{1}{3}x^3 + x - 1$. This function is invertible (you do not need to prove this).

1. Find $f^{-1}(-1)$.

   Solution. We must solve $-1 = \frac{1}{3}x^3 + x - 1$, or $0 = x(\frac{1}{3}x^2 + 1)$.
   Clearly, $x = 0$, so $f^{-1}(-1) = 0$. □

2. Find $(f^{-1})'(-1)$.

   Solution. Recall the rule for derivatives of inverses: $(f^{-1}(f(a)))' = \frac{1}{f'(a)}$. We know from above that $a = 0$, so we have $\frac{1}{f'(0)} = \frac{1}{(0)^2+1} = 1$. □

3. Find $L_{-1}^{f^{-1}}$.

   Solution. Apply the formula:

$$\begin{aligned} L_{-1}^{f^{-1}} &= f^{-1}(-1) + (f^{-1})'(-1)(x+1) \\ &= 0 + 1(x+1) \\ &= x + 1 \end{aligned}$$

□