MATH 137 Fall 2020: Practice Assignment 8

Q01. Find $\frac{dy}{dx}$ for $\arcsin\big(x^2y\big) + xy = 1$.

Solution. We implicitly differentiate with respect to x:

$$\begin{aligned}
\frac{d}{dx}\big(\arcsin\big(x^2y\big) + xy\big) &= \frac{d}{dx}(1) \\
\frac{1}{\sqrt{1-(x^2y)^2}}\frac{d}{dx}\big(x^2y\big) + \frac{d}{dx}(xy) &= 0 \\
\frac{2xy + x^2\frac{dy}{dx}}{\sqrt{1-x^4y^2}} + x\frac{dy}{dx} + y &= 0 \\
\frac{dy}{dx}\left(\frac{x^2}{\sqrt{1-x^4y^2}} + x\right) &= -y - \frac{2xy}{\sqrt{1-x^4y^2}} \\
\frac{dy}{dx} &= -\frac{2xy + y\sqrt{1-x^4y^2}}{x^2 + x\sqrt{1-x^4y^2}}
\end{aligned}$$

□

Q02. Find the equation of the tangent line at the point $(0, -1)$ to the curve defined by

$$xy + y^3 = \arctan(x) - 1.$$

Solution. We apply the point-slope form of a line: $y = y_0 + m(x - x_0)$. To find the slope, implicitly differentiate with respect to x:

$$\begin{aligned}
\frac{d}{dx}\big(xy + y^3\big) &= \frac{d}{dx}(\arctan x - 1) \\
x\frac{dy}{dx} + y + (3y^2)\frac{dy}{dx} &= \frac{1}{1+x^2} \\
\frac{dy}{dx}(x + 3y^2) &= \frac{1 - y(1+x^2)}{1+x^2} \\
\frac{dy}{dx} &= \frac{1 - y - x^2y}{(1+x^2)(x+3y^2)}
\end{aligned}$$

We can now find m by substituting $x = 0$ and $y = -1$:

$$\begin{aligned}
m &= \frac{1 - (-1) - (0)^2(-1)}{(1+(0)^2)((0) + 3(-1)^2)} \\
&= \frac{2}{3}
\end{aligned}$$

Therefore, the equation of the tangent line is $y = -1 + \frac{2}{3}(x - 0) = \frac{2}{3}x - 1$. □

Q03. Use the logarithmic differentiation to find $\frac{dy}{dx}$

(a) $y = x^{\sin x}$ with $x > 0$.

Solution. We may take the logarithm of both sides, then implicitly differentiate with respect to x:

$$\begin{aligned}\ln y &= \sin x \ln x \\ \frac{\mathrm{d}}{\mathrm{d}x}(\ln y) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin x \ln x) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x \ln x + \frac{\sin x}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{x}(x\cos x \ln x + \sin x)\end{aligned}$$ □

(b) $y = (2x)^{x^{1/3}}$.

Solution. Likewise,

$$\begin{aligned}\ln y &= x^{\frac{1}{3}} \ln 2x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{3x^{\frac{2}{3}}} \cdot \ln 2x + x^{\frac{1}{3}} \cdot \frac{1}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(\ln 2x + 3)}{3x^{\frac{2}{3}}}\end{aligned}$$ □

Q04. Find all critical points for the following functions.

(a) $f(x) = x + \frac{1}{x}$.

Solution. Calculate that $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$. Recall that critical points are defined as those where the derivative is zero or undefined. Since this is a rational function, we may say that all zeroes are zeroes of the denominator, and all undefined points are zeroes of the numerator. The numerator is zero at $x = \pm 1$, and the denominator is zero at $x = 0$.

Therefore, critical points occur at $x = -1, 0, 1$. □

(b) $f(x) = \frac{(x-1)^3}{(x+1)^4}$.

Solution. Likewise, the numerator is zero at $x = 1$, and the denominator at $x = -1$.

Therefore, critical points occur at $x = -1, 1$. □

Q05. Find the global maximum and minimum of $f(x) = 2\cos x + \sin 2x$ for $x \in [0, \frac{\pi}{2}]$.

Solution. Global extrema can either occur on the endpoints or on the open interval. We can calculate $f(0) = 2(1) + 0 = 2$ and $f(\frac{\pi}{2}) = 2(0) + 0 = 0$.

Local extrema on an open interval are given by points where $f'(x) = 0$. Taking the derivative, $f'(x) = -2\sin x + 2\cos 2x$. We can solve the trigonometric equation $0 = \cos 2x - \sin x$ using the double angle formula:

$$\begin{aligned}0 &= \cos 2x - \sin x \\ &= 1 - 2\sin^2 x - \sin x \\ &= 2\sin^2 x + \sin x - 1 \\ &= (2\sin x - 1)(\sin x + 1)\end{aligned}$$

So we must have either $\sin x = \frac{1}{2}$ or $\sin x = -1$. Limited to the domain $x \in (0, \frac{\pi}{2})$, our only solution is $x = \frac{\pi}{6}$. We check $f'(\frac{\pi}{6})$ and notice it is $2(\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \cong 2.6$.

Of these three, we have the global minimum 0 at $x = \frac{\pi}{2}$ and maximum $\frac{3\sqrt{3}}{2}$ at $x = \frac{\pi}{6}$. □

Q06. Let us use the Mean Value Theorem to compare the geometric and arithmetic means.

(a) Use the MVT to show that

$$\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}$$

for $0 < a < b$.

Proof. Consider the function $f(x) = \sqrt{x}$. Since $\sqrt{x}$ is both continuous and differentiable on $[a, b]$, we may apply the MVT. There exists a $c \in (a, b)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b-a} \\ \frac{1}{2\sqrt{c}} &= \frac{\sqrt{b} - \sqrt{a}}{b-a} \end{aligned}$$

Now, $c > a$ so $\frac{1}{2\sqrt{a}} > \frac{1}{2\sqrt{c}}$. We multiply through $b - a$, which is positive since $b > a$:

$$\begin{aligned} \frac{1}{2\sqrt{a}} &> \frac{1}{2\sqrt{c}} \\ &> \frac{\sqrt{b} - \sqrt{a}}{b-a} \\ \frac{b-a}{2\sqrt{a}} &> \sqrt{b} - \sqrt{a} \end{aligned}$$

completing the proof. □

(b) Use part (a) to show that, for $0 < a < b$, the geometric mean $\sqrt{ab}$ is always smaller than the arithmetic mean $\frac{1}{2}(a + b)$, that is, show that

$$\sqrt{ab} < \frac{a+b}{2}.$$

Proof. Manipulate the result from above, knowing that $\sqrt{a}$ is a positive number:

$$\begin{aligned} \sqrt{b} - \sqrt{a} &< \frac{b-a}{2\sqrt{a}} \\ 2\sqrt{a}\sqrt{b} - 2a &< b - a \\ 2\sqrt{ab} &< a + b \\ \sqrt{ab} &< \frac{a+b}{2} \end{aligned}$$

as desired. □

Q07. Show that the function $f(x) = 2x^5 + 2x + 1$ has exactly one root without sketching the graph of the function.

Hint: Assume there is more than 1 root and use the MVT to build a contradiction.

Proof. Notice that $f(0) = 1$ and $f(-1) = -3$. Since polynomials are continuous, we may apply the IVT. Therefore, a root exists on $x \in (-1, 0)$.

Suppose for a contradiction that $f(x) = 2x^5 + 2x + 1$ has more than one root. Then, by Rolle's Theorem, there exists some point where $f'(x) = 0$. However, $f'(x) = 10x^4 + 2$ which is positive for all real x. Therefore, by contradiction, $f(x)$ has at most one root. □

Q08. Let $f(x)$ be differentiable on (a, b) and $f'(x)$ be continuous on (a, b). Assume there are three points x_1, x_2, and x_3 with each $x_i \in (a, b)$ and with $x_1 < x_2 < x_3$ such that

$$f(x_1) < f(x_2) \quad \text{and} \quad f(x_2) > f(x_3).$$

Use the MVT and the IVT to show that there must be a point $c \in (x_1, x_3)$ such that $f'(c) = 0$.

Proof. Since f is differentiable on both $[x_1, x_2]$ and $[x_2, x_3]$, we may apply the MVT. As $f(x_1) < f(x_2)$, the mean slope is a positive value. Therefore, there exists a $c_1 \in (x_1, x_2)$ where $f'(c_1)$ is that positive value. As $f(x_2) > f(x_3)$, the mean slope is a negative value. Therefore, there also exists a $c_2 \in (x_2, x_3)$ where $f'(c_2)$ is that negative value.

We are also given that f' is continuous on $[x_1, x_3]$, so we may apply the IVT. From above, $f'(c_2) < 0 < f'(c_1)$, so there must exist a $c \in (c_1, c_2)$ where $f'(c) = 0$. Since $(c_1, c_2) \subsetneq (x_1, x_3)$, we may say that $c \in (x_1, x_3)$, completing the proof. □