MATH 137 Fall 2020: Practice Assignment 9

Q01. Find the intervals over which the following functions are increasing/decreasing.

(a) $f(x) = x^4 - 8x^2$

Solution. We take $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-4)(x+4)$. The critical points are $x = 0, \pm 4$. Since f' is an odd-degree polynomial with a positive leading coefficient and linear factors, we can say that f is decreasing on $(-\infty, -4) \cup (0, 4)$ and increasing on $(-4, 0) \cup (4, \infty)$. □

(b) $f(x) = \frac{1}{x^2-1}$

Solution. We take $f'(x) = -\frac{2x}{(x^2-1)^2} = -\frac{2x}{(x-1)(x+1)}$ and find critical points $x = 0, \pm 1$. Analyzing the signs of the factors of f':

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$-2x$	+	+	−	−
$(x-1)$	−	−	−	+
$(x+1)$	−	+	+	+
f'	+	−	+	−

Then, f is decreasing on $(-1, 0) \cup (1, \infty)$ and increasing on $(-\infty, -1) \cup (0, 1)$. □

(c) $f(x) = e^x + e^{-x+1}$

Solution. We have $f'(x) = e^x - e^{-x+1}$. This is defined on $\mathbb{R}$, so we solve $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \\ e^x &= e^{-x+1} \\ x &= -x + 1 \\ x &= \frac{1}{2} \end{aligned}$$

Therefore, our only critical point is at $x = \frac{1}{2}$. For large positive x, the e^x term dominates and for large negative x, the e^{-x} term dominates. It follows that f is decreasing on $(-\infty, \frac{1}{2})$ and increasing on $(\frac{1}{2}, \infty)$. □

(d) $f(x) = x^4 - 4x^3 + 16x - 7$

Solution. Taking the derivative, $f'(x) = 4x^3 - 12x^2 + 16 = 4(x+1)(x-2)^2$, and the critical points are $x = -1, 2$.

Since $(x-2)^2$ is always non-negative, it does not affect the sign of f'. From the sign of $(x+1)$, we can say f is increasing on $(-\infty, -1)$ and increasing on $(-1, \infty)$. □

Q02. Show that if f is increasing and differentiable on (a, b) then $f'(x) \geq 0$ for all $x \in (a, b)$.

Hint: You may wish to use the result

$$\text{If } g(x) > 0 \text{ for all } x \neq a \text{ and } \lim_{x \to a} g(x) = L, \text{ then } L \geq 0.$$

Proof. Let f be an increasing and differentiable function on (a, b), and let $x \in (a, b)$.

Since f is differentiable, $f'(x)$ exists and is equal to $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Since the limit exists, the one-sided limits exist and are equal. Consider the right-handed limit. Then, $h > 0$ and $x + h > x$. Because f is increasing, $f(x + h) > f(x)$ and $f(x+h) - f(x) > 0$. Therefore, the Newton quotient is positive for all h, so the limit, i.e., the derivative, is positive. □

Q03. Suppose f is a differentiable function that satisfies $f(1) = 3$ and $2 \le f'(x) \le 7$. Use the Bounded Derivative Theorem to find an interval for $f(3)$.

Solution. Since the lower bound of f' is 2, over a distance $3 - 1 = 2$, f can increase by at least 4. Likewise, as the upper bound of f' is 7, over a distance 2, f can increase by at most 14. Therefore, we have the range $f(3) \in [3 + 4, 3 + 14] = [7, 17]$. □

Q04. Assume f is a differentiable function on $\mathbb{R}$.

(a) Prove that if $|f'(x)| \le M$ for all $x \in \mathbb{R}$, then $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$. [Functions with this property are called *Lipschitz*].

Proof. Let f be differentiable and let x, y, and $M \ge 0$ be real numbers. Suppose $|f'(n)| \le M$, that is, $-M \le f'(n) \le M$ for all n.

Then, $f(y)$ is at most $f(x) + M|x - y|$ and at least $f(x) - M|x - y|$.

That is, $f(y) - f(x) \le \pm M|x - y|$, or, $|f(x) - f(y)| \le M|x - y|$. □

(b) Is the converse of part (a) true? Prove it or give a counterexample.

Solution. Yes. Suppose $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$. Then,

$$|f'(x)| = \lim_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} \le \frac{M|x + h - x|}{|h|} = M$$

since $|x|$ is continuous. □

Q05. Let $f(x) = \sqrt{x}$ and let $g(x) = 1 + \ln x$.

(a) Show that there is at least one point of intersection of f and g between e^2 and e^4.

Proof. Consider the function $h(x) = f(x) - g(x)$. Since h is composed of continuous functions, it is continuous on its domain ($x > 0$). Then, $h(e^2) = \sqrt{e^2} - 1 - \ln e^2 = e - 3$ and $h(e^4) = \sqrt{e^4} - 1 - \ln e^4 = e^2 - 5$.

As $e - 3 < 0$ and $e^2 - 5 > 0$, by the IVT, there exists a $c \in (e^2, e^4)$ where $h(c) = 0$, that is, $f(c) = g(c)$. □

(b) Show that there is exactly one point of intersection of f and g between e^2 and e^4. Call this point $x = b$.

Proof. Let $b_0, b_1 \in (e^2, e^4)$. Suppose for a contradiction that $h(b_0) = 0$ and $h(b_1) = 0$. Then, by the MVT, there exists some $c \in (b_0, b_1) \subsetneq (e^2, e^4)$ where $h'(c) = 0$. Now,

$$\begin{aligned} h'(c) &= f'(c) - g'(c) \\ 0 &= \frac{1}{2\sqrt{c}} - \frac{1}{c} \\ 0 &= \frac{\sqrt{c} - 2}{2c} \\ c &= 4 \end{aligned}$$

(since $0 \notin (b_0, b_1)$) but $4 \notin (e^2, e^4)$. Therefore, there cannot be a second point of intersection. □

(c) Show that for all $x > b$ we have $f(x) > g(x)$. That is, there are no more intersection points after $x = b$.

Proof. Notice from above that $h'(x) = 0$ only when $x = 4$. When $x > 4$, $h'(c) < 0$, and as h' is continuous on its domain, h is decreasing.

Since $b \in (e^2, e^4)$, we have $4 < e^2 < b$, h is decreasing for all $x > b$. Then, $h(b) > h(x) = f(x) - g(x)$, so $f(x) > g(x)$ for all $x > b$. □

Q06. Evaluate the following limits, you may use any method.

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x + x^2 - x}{\sin^2 x}$.

Solution. We evaluate the fraction and find that it is of the form $\frac{\tan 0 + 0^2 - 0}{\sin^2 0} = \frac{0}{0}$. Repeatedly applying l'Hôpital's rule:

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x + x^2 - x}{\sin^2 x} &= \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan x + x^2 - x\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^2 x\right)}\right|_{x=0} \\ &= \lim_{x\to 0} \frac{\sec^2 x + 2x - 1}{2\sin 2x} \\ &= \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^2 x + 2x - 1\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin 2x\right)}\right|_{x=0} \\ &= \lim_{x\to 0} \frac{2\sec^2 x \tan x + 2}{2\cos 2x} \\ &= \frac{0+2}{2(1)} \\ &= 1 \end{aligned}$$

□

(b) $\displaystyle\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

Solution. Simplify the fraction and apply l'Hôpital's Rule to forms $\frac{0}{0}$:

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1}\frac{x\ln x-(x-1)}{\ln x(x-1)} \\
&= \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x - x+1)}{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x(x-1))}\right|_{x=1} \\
&= \lim_{x\to 1}\frac{\ln x}{\ln x+\frac{x-1}{x}} \\
&= \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x+\frac{x-1}{x}\right)}\right|_{x=1} \\
&= \lim_{x\to 1}\frac{1}{x\frac{1}{x}+\frac{1}{x^2}} \\
&= \frac{1}{2}
\end{aligned}$$

□

(c) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$.

Solution. This is of the form 1^∞ so we take the logarithm:

$$\begin{aligned}
\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^{2x} &= \lim_{x\to\infty}2x\ln\left(1+\frac{1}{x}\right) \\
&= \lim_{x\to\infty}\frac{2\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \left.\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln\left(1+\frac{1}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}}\right|_{x=\infty} \\
&= \lim_{x\to\infty}\frac{-2\frac{1}{1+\frac{1}{x}}\frac{1}{x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty}2\frac{1}{1+\frac{1}{x}} \\
&= 2
\end{aligned}$$

□

Q07. Let $f(x)=x+\sin x\cos x$ and let $g(x)=f(x)e^{\sin x}$.

(a) Argue why $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}$ does not exist.

Proof. Note that $\frac{f(x)}{g(x)}=\frac{f(x)}{f(x)e^{\sin x}}=\frac{1}{e^{\sin x}}$. Since $\sin x$ is periodic and has no infinite limit, $\frac{1}{e^{\sin x}}$ oscillates between the values $\frac{1}{e}$ for $x=\frac{\pi+4k}{2}$ and e for $x=\frac{3\pi+4k}{2}$, $k\in\mathbb{Z}$, which are not equal.

Therefore, picking some sequence with those values, limit cannot exist. □

(b) Prove that $\displaystyle\lim_{x\to\infty}f(x)=\infty$ and $\displaystyle\lim_{x\to\infty}g(x)=\infty$.

Proof. Note that $\sin x \cos x \geq -1$ for all x. Then, $f(x) \geq x - 1$ for all x, but $x - 1$ diverges to infinity. Therefore, $\lim_{x\to\infty} f(x) = \infty$.

Now, $\sin x$ has range $[-1, 1]$, so $e^{\sin x}$ has range $[e^{-1}, e]$. Then, $g(x) \geq f(x)e^{-1}$, but we established that $f(x)$ diveres, so $\lim_{x\to\infty} g(x) = \infty$. ☐

(c) Prove that $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = 0$.

Proof. Take some derivatives to get $f'(x) = 1 + \cos 2x$ and

$$\begin{aligned} g'(x) &= f'(x)e^{\sin x} + f(x)e^{\sin x}\cos x \\ &= (1 + \cos 2x)e^{\sin x} + (x + \sin x \cos x)e^{\sin x}\cos x \\ &= e^{\sin x}(1 + \cos x + \frac{\sin 2x}{2}) + (e^{\sin x}\cos x)x \end{aligned}$$

Note that $0 \leq f'(x) \leq 2$ for all x, so $0 \leq \left|\frac{f'(x)}{g'(x)}\right| \leq 2$. The first term in $g'(x)$ is also clearly bounded. However, the second term is a bounded term multiplied by x, so it is unbounded. Therefore, $|g'(x)|$ can be made arbitrarily large. It follows by some squeeze theorem bullshit that $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = 0$. ☐

(d) Why is the above not a contradiction to l'Hôpital's Rule?

Answer. $f'(x)$ does not go to 0 or ∞, so the limit not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. ☐