MATH 137 Fall 2020: Practice Assignment on Chapter 5

Q01. Approximate $f(x) = x^{-1/2}$ with a Taylor polynomial of degree 2 centered at $x = 4$. Use Taylor's Theorem to get an upper bound on the error if $3.5 \le x \le 4.5$.

Solution. First calculate $f'(x) = -\frac{1}{2}x^{-3/2}$ and $f''(x) = \frac{3}{4}x^{-5/2}$. Then, $f(4) = 4^{-1/2} = \frac{1}{2}$, $f'(4) = -\frac{1}{2}4^{-3/2} = \frac{1}{16}$, and $f''(4) = \frac{3}{4}4^{-5/2} = \frac{3}{128}$. Then,

$$\begin{aligned} T_{2,4}(x) &= f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 \\ &= \frac{1}{2} + \frac{x-4}{16} + \frac{3(x-4)^2}{256} \end{aligned}$$

Taylor's Theorem gives the error $R_{2,4}(x) = \frac{f^{(3)}(c)}{3!}(x-4)^3$ for some $c \in [3.5, 4.5]$. The third derivative of f is $-\frac{15}{8}x^{-7/2}$. This is strictly increasing (i.e. $f^{(4)} > 0$) and negative (i.e. $f^{(3)} < 0$), so the maximum $|f^{(3)}(c)|$ is at $|f^{(3)}(3.5)| \approx 0.02338$.

Thus, $|R_{2,4}(x)| \le \left|\frac{f^{(3)}(3.5)}{6}(0.5)^3\right| \approx 4.87 \times 10^{-4}$. □

Q02. Approximate $f(x) = \ln(1 + 2x)$ with a Taylor polynomial of degree 3 centered at $x = 1$. Use Taylor's Theorem to get an upper bound on the error if $0.5 \le x \le 1.5$.

Solution. We have $f'(x) = \frac{2}{1+2x}$, $f''(x) = -\frac{4}{(1+2x)^2}$, and $f^{(3)}(x) = \frac{16}{(1+2x)^3}$. Calculate $f(1) = \ln 3$, $f'(1) = \frac{2}{3}$, $f''(1) = -\frac{4}{9}$, and $f^{(3)}(1) = \frac{16}{27}$. The Taylor polynomial is

$$\begin{aligned} T_{3,1}(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f^{(3)}(1)(x-1)^3 \\ &= \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3 \end{aligned}$$

Taylor's Theorem gives the error $R_{3,1}(x) = \frac{f^{(4)}(c)}{4!}(x-4)^3$ for some $c \in [0.5, 1.5]$. The fourth derivative of f is $-\frac{48}{(1+2x)^4}$. We can conclude that $|f^{(4)}|$ reaches its max at $c = 0.5$ through geometric argument, knowing the function is rational with one asymptote at $x = -\frac{1}{2}$ and no roots. We have $|f^{(4)}(0.5)| = 6$.

Thus, $|R_{3,1}(x)| \le \left|\frac{f^{(4)}(0.5)}{24}(0.5)^4\right| = 0.015625$. □

Q03. Here we approximate the value of $\ln 2$ in two ways.

(a) Find the degree 3 Taylor polynomial for $\ln(1 + x)$ centred at $x = 0$.

Solution. Let $f(x) = \ln(1 + x)$. Then we have $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, and $f^{(3)}(x) = \frac{2}{(1+x)^3}$. Evaluating at $x = 0$, we have $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, and $f^{(3)}(0) = 2$. Therefore, the Taylor polynomial $T_{3,0}(x)$ is

$$\begin{aligned} T_{3,0}(x) &= \frac{f^{(3)}(x)}{3!}x^3 + \frac{f''(x)}{2!}x^2 + f'(x)x + f(x) \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x \end{aligned}$$ □

(b) Use $x = 1$ in your polynomial from part (a) to approximate the value of $\ln 2$.

Solution. Plug and chug: $T_{3,0}(1) = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$. ☐

(c) Use $x = -\frac{1}{2}$ in your polynomial from part (a) to approximate the value of $\ln 2$. You will need to relate your answer to $\ln 2$ with log rules. Show that the upper bound on the error given by Taylor's Theorem is the same for your approximations from parts (b) and (c).

Solution. At $x = -\frac{1}{2}$, we have $f(x) = \ln\left(\frac{1}{2}\right) = -\ln 2$. Plugging and chugging, $T_{3,0}(-\frac{1}{2}) = \frac{1}{3}(-\frac{1}{8}) - \frac{1}{2}(\frac{1}{4}) - \frac{1}{2} = -\frac{2}{3}$. Therefore, our estimate is $\ln 2 \approx \frac{2}{3}$.

The error $|R_{3,0}|$ depends on the maximum value of $|f^{(4)}(x)| = \frac{6}{(1+x)^4}$. This value is decreasing everywhere, the maximum value is at $x = 0$ for $[0, 1]$ and $x = -\frac{1}{2}$ for $[-\frac{1}{2}, 0]$: $|f^{(4)}(0)| = 6$ and $|f^{(4)}(-\frac{1}{2})| = 96$.

Therefore, the error for part (b) is at least

$$|R_{3,0}(1)| \leq \frac{6}{4!}(1)^4 = 0.25$$

and the error above is at least

$$|R_{3,0}(-0.5)| \leq \frac{96}{4!}(0.5)^4 = 0.25$$ ☐

(d) Use a calculator to compare your approximations in part (b) and (c) with the actual value of $\ln 2$. Which is actually closer, and why does this make sense?

Solution. Calculator gives $\ln 2 \approx 0.693147$.

Part (b) estimated $\frac{5}{6} \approx 0.833333$ which an error of about -0.140186 and part (c) estimated $\frac{2}{3} \approx 0.666667$ which is off by 0.026480.

Part (c) was actually closer, and this makes sense because we are working closer to the center of the Taylor polynomial. ☐

Q04. Use Taylor's Theorem to find $n \in \mathbb{N}$ so that using $T_{n,0}(x)$ to approximate e^x at $x = 0.1$ has an error of at most 0.00001

Solution. Let $f(x) = e^x$. Recall that $f^{(n)}(x) = e^x$ for any $n \in \mathbb{N}$. Since e^x is increasing everywhere, the maximum on $[0, 0.1]$ will be at $x = 0.1$ Then, Taylor's Theorem gives

$$|R_{n,0}(x)| \leq \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1}$$

but we have $e^{0.1}$ stuck in there. We can give an upper bound by doing some shenanigans. $e^0.1$ is the tenth root of e. This is clearly less than the tenth root of 3. Now, $1.1^{10} \approx 2.6$ and $1.2^{10} \approx 6.2$, so we give $\sqrt[10]{e} \leq 1.2$. Then,

$$\begin{aligned} |R_{n,0}(x)| &\leq \frac{e^x}{(n+1)!}(0.1)^{n+1} \\ 0.00001 &\leq \frac{1.2}{(n+1)!}(0.1)^{n+1} \end{aligned}$$

and we find by Pain and Agony™ that we need $n \geq 3$. ☐

Q05. Let us revisit Newton's Method one more time using Taylor's Theorem. Suppose we are approximating the root r of the function f. Recall that from an initial approximation x_1, we obtained the successive approximations using the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Use Taylor's Theorem (or inequality) with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \leq M$, $|f'(x)| \geq K$ for all $x \in I$, then

$$|x_{n+1} - r| \leq \frac{M}{2K}|x_n - r|^2.$$

(Note that this says that if the error at stage n is at most 10^{-m}, then the error at stage $n+1$ is at most $\frac{M}{2K}10^{-2m}$, or in other words, that successive iterations are accurate to approximately twice as many decimal places!)

Proof. We follow the instructions and do as we're told. Then, we have $f(r) = 0$, $f(r) = T_{1,x_n}(r) + R_{1,x_n}(r)$, and $T_{1,x_n}(r) = f'(r)(r - x_n) + f(r)$.

Substituting, $0 = f'(r)(r - x_n) + f(r) + R_{1,x_n}(r)$.

Then, we have $R_{1,x_n}(r) = -f(r) + f'(r)(x_n - r)$. We want $\frac{f(x_n)}{f'(x_n)}$ so we divide through by $f'(x_n)$ to get $\frac{R_{1,x_n}(r)}{f'(x_n)} = x_n - \frac{f(x_n)}{f'(x_n)} - r = x_{n+1} - r$. Therefore, $|\frac{R_{1,x_n}(r)}{f'(x_n)}| = |x_{n+1} - r|$. Since $f'(x_n) \geq K$, we have $|x_{n+1} - r| \leq \frac{1}{K}|R_{1,x_n}(r)|$.

We can finally apply Taylor's Theorem and get that $|R_{1,x_n}(r)| = \frac{f''(c)}{2}|x_n - r|^2$. We know that $f''(c) \leq M$ so $|R_{1,x_n}(r)| \leq \frac{M}{2}|x_n - r|^2$.

Combining these, we have $|x_{n+1} - r| \leq \frac{M}{2K}|x_n - r|^2$. $\square$