MATH 137 Fall 2019: Practice Final Exam

Multiple Choice

MC01. $\lim_{x\to 3} \ln|x-3| =$

(a) 0.
(b) ∞.
(c) $\boxed{-\infty.}$ *Vertical asymptote*
(d) None of the above.

MC02. If f is continuous on $[a,b]$ and differentiable on (a,b), then

(a) $\boxed{\text{for any } x_1, x_2 \in (a,b) \text{ where } x_1 < x_2, \text{ exists } c \in (x_1,x_2) \text{ so that } f'(c) = \frac{f(x_1)-f(x_2)}{x_1-x_2}.}$
Statement of the MVT
(b) $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$.
(c) $f'(x)$ is continuous on (a,b).
(d) None of the above.

MC03. $\lim_{x\to 0} \frac{\cos x + \sin x - 1}{x} =$

(a) -1.
(b) 0.
(c) 2.
(d) $\boxed{\text{None of the above.}}$ *And is equal to 1*

MC04. If $\lim_{x\to a} f(x) = L \in \mathbb{R}$ and $\{f(x_n)\}$ is a sequence such that $x_n \to a$ as $n \to \infty$, and $x_n \neq a$ for all $n \in \mathbb{N}$, then

(a) $\lim_{n\to\infty} f(x_n)$ does not exist.
(b) f is continuous at $x = a$.
(c) $\boxed{\lim_{n\to\infty} f(x_n) = L.}$ *Statement of SCL*
(d) None of the above.

MC05. For a function f and $a \in \mathbb{R}$, if $f''(a) = 0$ and $f'(a) = 0$, then

(a) $x = a$ is a point of inflection for f.
(b) $\boxed{x = a \text{ is a critical point for } f.}$ *By definition*
(c) f cannot have a local maximum at $x = a$.
(d) None of the above.

True/False

TF06. For $a \in \mathbb{R}$, $|x-a| \le 1$ defines a closed interval of length 1.
False. *The interval has length 2*

TF07. $f(x) = 3x^4 - 2x - 1$ haas a root on $[0,1]$.
True. *Notice that $f(1) = 0$*

TF08. If $f'(x) = \cos x$ then $f(x) = \sin x$.
False. *Missing constant of integration*

TF09. Let $a_n = f(n)$ where f is a continuous function defined on $\mathbb{R}$. If $\lim_{n\to\infty} a_n = L$ then $\lim_{x\to\infty} f(x) = L$.
False. *Let* $f(n+\frac{1}{2}) = L+1$ *for all* n

TF10. If f is not differentiable at $x = a \in \mathbb{R}$, then for $k \in \mathbb{R}$, $g(x) = f(x) + k$ is not differentiable at $x = a$
True. *The limit does not exist*

Short Answer

SA01. For $f(x) = \ln(e + x)$, find $L_0^f(x)$.

Solution. We know $f(0) = \ln e = 1$ and $f'(x) = \frac{1}{e+x}$ so $f'(0) = \frac{1}{e}$. Therefore,

$$L_0^f(x) = 1 + \frac{1}{e}x$$

□

SA02. If f is a differentiable function such that $f(0) = 1$ and $f'(x) \in [1, 5]$ for all $x \in \mathbb{R}$, use the Bounded Derivative Theorem to write down an interval that $f(3)$ must lie in.

Solution. Apply BDT: $f(3) \in [1 + 1(3), 1 + 5(3)] = [4, 16]$. □

SA03. Give an example of a differentiable function f that is concave up everywhere, but $f''(0)$ does not exist.

Solution. Let $f(x) = \begin{cases} x^2 & x \geq 0 \\ x^4 & x < 0 \end{cases}$.

At $x = 0$, the function remains differentiable since both one-sided limits and derivatives are 0. However, $f''(0)$ does not exist since the one-sided derivatives do not agree. For all points other than $x = 0$, $f''(x) > 0$, so f is concave up for positive and negative x. In fact, f is concave up everywhere. □

SA04. Give an example of a function f that is differentiable on $(0, 1)$, both $f(0)$ and $f(1)$ are defined, but the Mean Value Theorem cannot be applied to f.

Solution. We are given the hypotheses of MVT except continuity. Make f discontinuous:

$$f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

□

SA05. If $f(3) = 1$ and $f'(3) = \pi$, find $(f^{-1})'(1)$.

Solution. We know that $f^{-1}(1) = 3$. Then, by the IFT, $(f^{-1})'(1) = \frac{1}{f'(3)} = \frac{1}{\pi}$. □

Long Answer

LA01. Find each of the following sequence limits, if they exist. If they do not exist, prove it.

(a) $\lim_{n\to\infty} \frac{\sin n\pi}{\sin n}$

Solution. Recall that $\sin n\pi = 0$ for all n. Then, $\frac{\sin n\pi}{\sin n} = 0$ so the limit is 0. □

(b) $\lim\limits_{n\to\infty} \frac{\sin n}{n^2+1}$

Solution. We know that $-1 \leq \sin n \leq 1$, so we have

$$-\frac{1}{n^2+1} \leq \frac{\sin n}{n^2+1} \leq \frac{1}{n^2+1}$$

Now, both of these converge to 0, so by the Squeeze Theorem, the limit is 0. □

(c) $\lim\limits_{n\to\infty} \frac{n^3+n+1}{3n^3+n^2}$

Solution. Divide through by n^3:

$$\begin{aligned}\lim_{n\to\infty} \frac{n^3+n+1}{3n^3+n^2} &= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}+\frac{1}{n^3}}{3+\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1+0+0}{3+0} \\ &= \lim_{n\to\infty} \frac{1}{3}\end{aligned}$$ □

LA02. Prove that if $\{a_n\}$ and $\{b_n\}$ are sequences such that $\{a_n\}$ is bounded and $b_n \to 0$ as $n \to \infty$, then $\lim\limits_{n\to\infty}(a_n b_n) = 0$.

Proof. Since a_n is bounded, we have $x = \sup a_n$ and $y = \inf a_n$ such that $x \leq a_n \leq y$ for all n. Multiplying through by b_n, we have $xb_n \leq a_n b_n \leq yb_n$. As b_n converges to 0, so too do xb_n and yb_n by the arithmetic rules.

Therefore, by the Squeeze Theorem, the limit is 0. □

LA03. For each of the following functions, compute $f'(x)$ using any method. You do not need to simplify your answers.

(a) $f(x) = x^2 e^x \ln x$

Solution. Repeatedly apply the product rule:

$$\begin{aligned}f'(x) &= 2xe^x \ln x + x^2(e^x \ln x)' \\ &= 2xe^x \ln x + x^2(e^x \ln x + \frac{e^x}{x})\end{aligned}$$ □

(b) $f(x) = \tan(\cos x)$

Solution. Apply the chain rule:

$$\begin{aligned}f'(x) &= \sec^2(\cos x)(\cos x)' \\ &= -\sec^2(\cos x)\sin x\end{aligned}$$ □

LA04. (a) Find y' if $\ln x + \ln y = xy$.

Solution. Implicitly differentiate with respect to x:

$$\begin{aligned}\frac{1}{x} + \frac{y'}{y} &= xy' + y \\ y'(\frac{1}{y} - x) &= y - \frac{1}{x} \\ y' &= \frac{y - \frac{1}{x}}{\frac{1}{y} - x}\end{aligned}$$ □

(b) Find $\frac{dy}{dx}$ if $y = (\sin x)^{\ln x}$ for $0 < x \leq \pi$.

Solution. Taking the logarithm of both sides, $\ln y = \ln x \ln(\sin x)$. Then, implicitly differentiating with respect to x:

$$\frac{y'}{y} = \frac{\ln(\sin x)}{x} + \frac{\ln x \cos x}{\sin x}$$
$$\frac{dy}{dx} = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \cot x\right)$$
□

LA05. For $f(x) = (x-1)|x+2| - 3$, determine all global extrema on the interval $[-3, 0]$, if they exist.

Solution. For $x < -2$, $f(x) = -(x-1)(x+2) - 3 = -x^2 - x - 1$ and $f'(x) = -2x - 1$. Likewise, if $x > -2$, $f(x) = (x-1)(x+2) - 3 = x^2 + x - 5$ and $f'(x) = 2x + 1$.

The critical points are at $x = -2$ (undefined) and $x = -\frac{1}{2}$ (zero). Testing the function values here and at the endpoints, we have $f(-3) = -7$, $f(-2) = -3$, $f(-\frac{1}{2}) = -\frac{21}{4}$, and $f(0) = -5$.

Therefore, the global extrema are at $(-3, -7)$ and $(-2, -3)$. □

LA06. (a) State the Intermediate Value Theorem for a function f.

Theorem. If f is continuous on $[a, b]$ and $f(a) < f(b)$, then for any value $k \in (f(a), f(b))$ there exists a $c \in (a, b)$ where $f(c) = k$. Likewise, if $f(b) < f(a)$, then for any value $k \in (f(b), f(a))$, there exists a $c \in (a, b)$ where $f(c) = k$.

(b) Find an interval of length at most 1 that contains a root of $f(x) = x^3 + 3x + 1$.

Solution. We know that $f(0) = 1$ and $f(-1) = -3$, so by the IVT, a root exists on $(0, 1)$. □

(c) Using $x_1 = 0$, perform two iterations of Newton's Method to find x_2 and x_3 to approximate the root of $f(x) = x^3 + 3x + 1$.

Solution. We have $f'(x) = 3x^2 + 3$ Then, $x_1 = 0$ and $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{f(0)}{f'(0)} = -\frac{1}{3}$. Again, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -\frac{1}{3} - \frac{-1/27}{10/3} = -\frac{29}{90}$.

Therefore, $f(-\frac{29}{90}) \approx 0$. □

LA07. Let $f(x) = \ln(x^2 + 1)$.

(a) Determine the intervals of increase/decrease for f.

Solution. We have $f'(x) = \frac{2x}{x^2+1}$. Since $x^2 + 1$ is always positive, $f'(x)$ has the same sign as x. Therefore, f is increasing for positive x and decreasing for negative x. □

(b) Determine the intervals of concavity for f.

Solution. Taking the derivative from above, $f''(x) = -\frac{2(x-1)(x+1)}{(x^2+1)^2}$. The denominator is always positive so this is well-defined. It is zero at $x = \pm 1$, and from a sign analysis, we determine that $f''(x)$ is positive on $(-1, 1)$ so f is concave up, and that $f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ so f is concave down. □

LA08. Prove that if f is a differentiable function with no critical points, then it can have at most one real root.

Proof. Let f be a differentiable function with no critical points. Suppose for a contradiction that f contains more than one real root, namely, a and b. Then, by Rolle's Theorem, since $f(a) = 0$ and $f(b) = 0$, there exists a point $c \in (a, b)$ such that $f'(c) = 0$, i.e., c is a critical point of f. However, f has no critical points.

Therefore, f has at most one real root. $\square$

LA09. In each case, compute the limit using any method.

(a) $\lim\limits_{x \to 1^+} (\ln x)^{x-1}$

Solution. This is indeterminate of the form 0^0. We can rewrite the quantity in the limit as $e^{\ln((\ln x)^{x-1})} = e^{(x-1)\ln(\ln x)}$. Since e^x is continuous, we can push through the limit and consider only the limit

$$\lim_{x \to 1^+} (\ln x)^{x-1} = e^{\lim_{x \to 1^+} (x-1)\ln(\ln x)}$$

We now have the form $0 \cdot -\infty$. We rewrite as $-\frac{\infty}{\infty}$ and apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x \to 1^+} (x-1)\ln(\ln x) &= \lim_{x \to 1^+} \frac{\ln(\ln x)}{\frac{1}{x-1}} \\
&= \lim_{x \to 1^+} \frac{\frac{1}{x \ln x}}{-\frac{1}{(x-1)^2}} \\
&= \lim_{x \to 1^+} \frac{(x-1)^2}{x \ln x} \\
&= \lim_{x \to 1^+} \frac{(x-1)^2}{x \ln x} && (\text{of the form } \frac{0}{0}) \\
&= \lim_{x \to 1^+} \frac{2(x-1)}{1 + \ln x} \\
&= \frac{0}{1} = 0
\end{aligned}$$

Now, $e^0 = 1$, so the limit is equal to 1. $\square$

(b) $\lim\limits_{x \to 0^+} (\sqrt{x})^{\frac{1}{3\sqrt{x}}}$

Solution. The limit is of the form 0^∞, so we can say that it is equal to 0. $\square$

(c) $\lim\limits_{x \to 0^+} (1 + \sqrt{x})^{\frac{1}{3\sqrt{x}}}$

Solution. The limit is indeterminate and of the form 1^∞. Doing our good ole' trickery, we rewrite as $e^{\frac{\ln(1+\sqrt{x})}{3\sqrt{x}}}$, and after pushing through, we have a form $\frac{0}{0}$ so we can apply l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x \to 0^+} \frac{\ln\left(1 + \sqrt{x}\right)}{3\sqrt{x}} &= \lim_{x \to 0^+} \frac{\frac{1}{2\sqrt{x}(1+\sqrt{x})}}{\frac{3}{2\sqrt{x}}} \\
&= \lim_{x \to 0^+} \frac{1}{3(1 + \sqrt{x})} \\
&= \frac{1}{3}
\end{aligned}$$

We can conclude the limit is $\sqrt[3]{e}$. $\square$

LA10. Find values of a and b so that f is differentiable everywhere, where

$$f(x) = \begin{cases} \sin ax & x \geq 0 \\ x^2 + 2x + b & x < 0 \end{cases}$$

Solution. For f to be differentiable, the one-sided derivatives must agree. These are $a \cos ax$ and $2x + 2$. At $x = 0$, these are equal to a and 2, so $a = 2$ Differentiability implies continuity, so we check the one-sided limits. Then, $\sin 0 = (0)^2 + 2(0) + b$, and we conclude $b = 0$.

Therefore, $(a, b) = (2, 0)$. □

LA11. (a) Prove that if $f'(x) = g'(x)$ for all x in some open interval I, then there exists $k \in \mathbb{R}$ so that $f(x) = g(x) + k$ for all $x \in I$.

Proof. This follows directly from the Constant Function Theorem. □

(b) Use part (a) to prove that if $f'(x) - g'(x) = 2x$ on I, then $f(x) = g(x) + x^2 + k$ for all $x \in I$ for some $k \in \mathbb{R}$.

Proof. Notice that $f'(x) = g'(x) + 2x = (g(x) + x^2)'$.
Then, from (a), $f(x) = g(x) + x^2 + k$. □

LA12. Consider the function $f(x) = \ln(1 + x)$.

(a) Find the second-degree Taylor polynomial for f centred at $x = 0$, $T_{2,0}(x)$.

Solution. We have that $f'(x) = \frac{1}{1+x}$ and $f''(x) = -\frac{1}{(1+x)^2}$. Then, $f(0) = 0$, $f'(0) = 1$, and $f''(0) = -1$. Now, applying the formula,

$$\begin{aligned} T_{2,0}(x) &= \frac{f''(0)}{2}x^2 + f'(0)x + f(0) \\ &= -\frac{1}{2}x^2 + x \end{aligned}$$

□

(b) Use $T_{2,0}$ to approximate $\ln 2$.

Solution. From above, evaluate $\ln 2 = \ln(1 + 1) \approx T_{2,0}(1) = \frac{1}{2}$. □

(c) Use Taylor's Theorem to write down what $f(x) - T_{2,0}(x)$ is equal to (in terms of x and c) for $x > 0$.

Solution. For some $c \in (0, 1)$, we have that $f(x) - T_{2,0}(x) = R_{2,0} = \frac{f^{(3)}(c)}{3!}x^3$. We can calculate $f^{(3)}(x) = \frac{2}{(1+x)^3}$, so we can expand this as

$$R_{2,0}(x) = \frac{x^3}{3(1 + c)^3}$$

□

(d) Find an upper bound on the error in your approximation in part (b).

Solution. From above, $R_{2,0}(1) = \frac{1}{3(1+c)^3}$. This is always positive and maximized when $c = 0$, so $R_{2,0}(1) \leq \frac{1}{3}$. □

(e) Is the estimate in part (b) an over or under estimate?

Solution. From above, the difference is positive, so it is an underestimate. □

(f) Give an interval that $\ln 2$ must lie in, be as specific as possible.

Solution. From part (d), it lies in $[\frac{1}{2}, \frac{1}{2} + \frac{1}{3}] = [\frac{1}{2}, \frac{5}{6}]$. □