**MATH 137 Fall 2020: Practice Midterm 1**

## Multiple Choice

**MC01**. For the sequence $\{a_n\}$ where $a_1 = 1$, $a_n = \sqrt{6 + a_{n-1}}$ for $n \geq 2$. The value of $\lim_{n\to\infty} a_n$ is

(a) $-3$

(b) $-2$

(c) $2$

(d) $\boxed{\text{None of the above.}}$ *$a_2 = \sqrt{7} > 2$ and $\{a_n\}$ is increasing*

**MC02**. $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1}$

(a) $= 1$.

(b) $\boxed{= \frac{1}{2}.}$ *Translate to* $\displaystyle\lim_{y\to 0} \frac{\sin y}{y^2 + 2y} = \lim_{y\to 0} \frac{\sin y}{y} \cdot \frac{1}{y+2}$

(c) Does not exist.

(d) None of the above.

**MC03**. If $f$ is not continuous at $x = 2$ then it must be the case that

(a) $f(2) \geq 0$.

(b) $f(2)$ is undefined.

(c) $f(2)$ is defined.

(d) $\boxed{\text{None of the above.}}$ *For examples,* $f(x) = \dfrac{1}{x-2}$ *and* $f(x) = \begin{cases} 0 & x \neq 2 \\ -1 & x = 2 \end{cases}$

**MC04**. The sequence defined by $a_n = \dfrac{n^2+1}{n+3}$

(a) converges.

(b) is non-increasing.

(c) $\boxed{\text{is bounded below.}}$ *Notice that $a_n \to \infty$*

(d) None of the above.

**MC05**. If $f(x) = 7$ for all $x \in \mathbb{R}$ then $f'(x)$

(a) $\boxed{\text{exists for all } x \in \mathbb{R}.}$ *And is equal to 0*

(b) is not continuous for all $x \in \mathbb{R}$.

(c) $= 1$.

(d) None of the above.

**True/False**

**TF06**. Three functions, $f$, $g$ and $h$, are defined on an open interval $I$ containing $x = a$. If for each $x \in I$, $g(x) < f(x) < h(x)$ and $\lim_{x\to a^+} g(x) = L = \lim_{x\to a^+} h(x)$, then $\lim_{x\to a} f(x) = L$.

False. If $\lim_{x\to a^-} g(x) \neq \lim_{x\to a^-} h(x)$, then $\lim_{x\to a^-} f(x)$ can be any value between (or undefined).

**TF07**. The Fundamental Trigonometric Limit tells us that if $\theta$ is small, then $\cos\theta \approx \theta$.

False. This is true for $\sin\theta$.

**TF08**. If $f$ is continuous on $\mathbb{R}$ and $f(0) > 0$ then there exists $\delta > 0$ so that $f(x) > 0$ for all $x \in (0, \delta)$.

True. This follows from the $\epsilon$–$\delta$ definition and that $\lim_{x\to 0} f(x) = f(0) > 0$.

**TF09**. $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ for all $p \in \mathbb{R}$.

True. For positive $p$, see course notes. For negative $p$, we have $\frac{1}{x^q e^x}$ for positive $q$, which converges to 0. For zero $p$, $\frac{1}{e^x}$ converges to 0.

**TF10**. If $f(a)$ exists, then $f'(a)$ exists too.

False. Consider for example $f(x) = \lfloor x \rfloor$, which is defined but is not continuous at $x = 2$.

**Short Answer**

**SA01**. For the function

$$f(x) = \begin{cases} 1 + \sin x & x < 0 \\ \cos x & 0 \leq x \leq \pi \\ \sin x & \pi < x \end{cases}$$

determine

(a) $\lim_{x\to 0} f(x)$, or write DNE if it does not exist.

*Solution.* From below, $\lim_{x\to 0^-}(1+\sin x) = 1+\sin 0 = 1$. From above, $\lim_{x\to 0^+}\cos x = \cos 0 = 1$. Since the one-sided limits agree, the limit exists and is $\boxed{1}$. □

(b) $\lim_{x\to \pi} f(x)$, or write DNE if it does not exist.

*Solution.* From below, $\lim_{x\to \pi^-}\cos x = \cos\pi = 1$. From above, $\lim_{x\to \pi^+}\sin x = \sin\pi = 0$. Since the one-sided limits do agree, the limit $\boxed{\text{DNE}}$. □

**SA02**. Write all solutions to $|x - 1| = |2x|$

*Solution.* For $x > 1$: $x - 1 = 2x \implies x = -1$.
For $0 < x < 1$: $-(x - 1) = 2x \implies x = \frac{1}{3}$.
For $x < 0$: $-(x - 1) = -(2x) \implies x = -1$.

Therefore, $\boxed{x \in \left\{-1, \frac{1}{3}\right\}}$. □

**SA03**. Give an example of a function such that the Extreme Value Theorem does not apply to it on the interval $[0, 5]$.

*Solution.* Let $\boxed{f(x) = \frac{1}{x-2}}$.

There is a vertical asymptote at $x = 2$, which breaks the Extreme Value Theorem. □

**SA04**. State the formal $\epsilon$–$\delta$ definition of what it means for $\lim_{x\to a} f(x) = L$.

*Solution.* For all $\epsilon > 0$, there exists a $\delta > 0$, such that for every $x$ in the domain of $f$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. That is,

$$\boxed{\forall\epsilon > 0, \exists\delta > 0, \forall x, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon}$$ □

**Long Answer**

**LA01**. Use the $\epsilon$–$N$ definition of the limit of a sequence to show $\lim_{n\to\infty} \frac{n+1}{3n+2} = \frac{1}{3}$.

*Proof.* Let $a_n = \frac{n+1}{3n+2}$ and $\epsilon > 0$. We must find $N$ so $n \geq N$ implies $\left|a_n - \frac{1}{3}\right| < \epsilon$.

Because $a_n - \frac{1}{3} = \frac{3n+3-3n-2}{9n+6} = \frac{1}{9n+6}$, it suffices to show $\left|\frac{1}{9n+6}\right| < \epsilon$. Since $9n+6$ is positive for all positive $n$, we may drop the absolute value bars.

Let $N = \frac{1}{9\epsilon}$. Then,

$$\begin{aligned} n \geq N &\implies n \geq \frac{1}{9\epsilon} \\ &\implies 9n \geq \frac{1}{\epsilon} \\ &\implies 9n + 6 > \frac{1}{\epsilon} \\ &\implies \frac{1}{9n+6} < \epsilon \end{aligned}$$

exactly as desired.

Therefore, by the $\epsilon$–$N$ definition of a limit of a sequence, $\lim_{n\to\infty} a_n = \frac{1}{3}$. □

**LA02**. Compute the following sequence limits, or show that they do not exist.

(a) $\lim_{n\to\infty} \frac{\cos n}{n}$

*Proof.* We propose that the limit is 0 and prove it. Recall that $-1 \leq \cos n \leq 1$ for all $n$. Then, for positive $n$, $-\frac{1}{n} < \frac{\cos n}{n} < \frac{1}{n}$.

Trivially, $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$. The limits agree and $\frac{\cos n}{n}$ is bounded by them above and below.

Therefore, by the squeeze theorem, $\frac{\cos n}{n}$ also converges to 0. □

(b) $\lim_{n\to\infty} \frac{2n^2-n-1}{5n^2+n-3}$

*Proof.* Recall that for any rational function $\frac{f(x)}{g(x)}$, if $\deg f = \deg g$, the limit at infinity is the ratio of the leading coefficients.

Therefore, the limit is $\frac{2}{5}$. □

**LA03**. Consider the recursive sequence $a_1 = 5$ and $a_{n+1} = \frac{a_n + 1}{3}$ for $n \geq 1$.

(a) Prove that the sequence is decreasing and is bounded below by 0.

*Proof.* We prove by induction of the sentence $0 < a_{n+1} < a_n$ on $n$.

For the base case, notice that $a_1 = 5$ and $a_2 = \frac{5+1}{3} = 2$. We have $0 < a_2 < a_1$.

Now, suppose that $0 < a_{k+1} < a_k$ for some $k$. Then,

$$\begin{aligned} 1 < a_{k+1} + 1 &< a_k + 1 \\ \frac{1}{3} < \frac{a_{k+1}+1}{3} &< \frac{a_k+1}{3} \\ 0 < a_{k+2} &< a_{k+1} \end{aligned}$$

as desired. Therefore, by induction, $0 < a_{n+1} < a_n$ for all $n$, that is, $a_n$ is decreasing and bounded below by 0. □

(b) Prove that the sequence converges and find its limit.

*Proof.* Because $a_n$ is non-increasing and bounded below, the limit exists and is equal to $L$ by the monotone convergence theorem.

Recall that if $a_n \to L$, then $a_{n+1} \to L$. Then,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\ L &= \lim_{n\to\infty} \frac{a_n+1}{3} \\ L &= \frac{\lim_{n\to\infty} a_n + 1}{3} \\ L &= \frac{L+1}{3} \\ L &= \frac{1}{2} \end{aligned}$$

□

**LA04**. Use the $\epsilon$–$\delta$ definition of the limit of a function to show that $\lim_{x\to 2} \left(x^2 + 2x - 3\right) = 5$.

*Proof.* Let $\epsilon > 0$. We must find $\delta$ so that $0 < |x-2| < \delta$ implies $\left|(x^2+2x-3) - 5\right| = |x^2 + 2x - 8| < \epsilon$.

We can limit $\delta$ by having it equal $\min(\{\frac{\epsilon}{7}, 1\})$. Then, when $|x-2| < \delta$ we have $|x+4| < 7$.

We now have $|x-2| < \delta \leq \frac{\epsilon}{7}$ and $|x+4| < 7$. Multiplying,

$$\begin{aligned} |x-2| \cdot |x+4| &< \frac{\epsilon}{7} \cdot 7 \\ |(x-2)(x+4)| &< \epsilon \\ |x^2 + 2x - 8| &< \epsilon \end{aligned}$$

Therefore, by the $\epsilon$–$\delta$ definition of the limit of a function, $\lim_{x\to 2}\left(x^2+2x-3\right)=5$. □

**LA05**. Compute the following function limits, if possible. If the limit does not exist, prove it.

(a) $\lim_{x\to 0}\dfrac{5x^2-3x}{2x^3-x^2}$

*Solution.* Recall the continuity of polynomials and quotients. It follows that all rational functions $\frac{p(x)}{q(x)}$ are continuous at any $x=a$ so long as $q(a)\neq 0$.

Let $f(x)=\frac{5x^2-3}{2x^3-x^2}$. At $x=0$, we have $2x^3-x^2=0$. Therefore, $f$ is not continuous at $x=0$ and we analyze the one-sided limits to determine the type of discontinuity.

First, notice that we may factor as $f(x)=\frac{1}{x^2}\cdot\frac{5x-3}{2x-1}$.

Consider the sequence $a_n=\frac{1}{n}$, a sequence which converges to 0. We have

$$\lim_{n\to\infty} f(a_n)=\lim_{n\to\infty} n^2\left(\frac{\frac{5}{n}-3}{\frac{2}{n}-1}\right)=\infty$$

By the sequential characterization of limits, if the limit of $f(a_n)$ does not exist, so too does the limit $f$ at $x=0$.

Therefore, $\lim_{x\to 0}\dfrac{5x^2-3x}{2x^3-x^2}$ does not exist. □

(b) $\lim_{x\to 0}\dfrac{3x^2-1}{x^2-x+1}$

*Proof.* Recall again the continuity of rational functions.

Here, the denominator is $(0)^2-(0)+1=1\neq 0$, therefore we may simply evaluate the function at $x=0$. This is $\frac{0-1}{0-0+1}=-1$. □

(c) $\lim_{x\to\infty}\dfrac{\ln x^2-\ln x}{x^2-x}$

*Solution.* Simplify:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x^2-\ln x}{x^2-x} &= \lim_{x\to\infty}\frac{2\ln x-\ln x}{x(x-1)} && \text{by logarithm laws}\\ &= \lim_{x\to\infty}\frac{\ln x}{x}\cdot\lim_{x\to\infty}\frac{1}{x-1} && \text{by limit laws}\\ &= 0\cdot 0 && \text{by FLL} \quad \square\end{aligned}$$

**LA06**. Given the function

$$f(x)=\begin{cases} k^2x+5 & x\leq -2\\ x^2+k & x>-2\end{cases}$$

If $f(x)$ is continuous at $x=-2$, determine all value(s) for $k$.

*Solution.* Recall that for $f$ to be continuous at $x = -2$, $\lim_{x \to -2} f(x) = f(-2)$.

This limit exists if and only if the one-sided limits,

$$\lim_{x \to -2^-} f(x) = -2k^2 + 5 \qquad \text{and} \qquad \lim_{x \to -2^+} f(x) = 4 + k$$

agree as $x$ approaches $-2$ from above and below. That is,

$$\begin{aligned} -2k^2 + 5 &= 4 + k \\ 0 &= 2k^2 + k - 1 \end{aligned}$$

which is a quadratic in $k$. Factoring, $0 = (2k-1)(k+1)$. Therefore, $k \in \left\{-1, \frac{1}{2}\right\}$. □

**LA07**. Determine, with justification, all vertical asymptotes of the function

$$f(x) = \frac{x+3}{|x^2 - 2x - 15|}.$$

*Solution.* Recall the continuity of polynomials and quotients. It follows that all rational functions $\frac{p(x)}{q(x)}$ are continuous at any $x = a$ so long as $q(a) \neq 0$.

Factoring, $\left|x^2 - 2x - 15\right| = |(x-5)(x+3)| = |x-5| \cdot |x+3|$. This is zero at $x = -3, 5$. We consider these two options:

- Consider $x = -3$. The limit from below is:

  $$\lim_{x \to -3^-} \frac{x+3}{|x-5| \cdot |x+3|} = \lim_{x \to -3^-} \frac{x+3}{-(x-5) \cdot -(x+3)} = \lim_{x \to -3^-} \frac{1}{x-5} = \frac{1}{8}$$

  and the limit from above is:

  $$\lim_{x \to -3^+} \frac{x+3}{|x-5| \cdot |x+3|} = \lim_{x \to -3^+} \frac{x+3}{-(x-5)(x+3)} = \lim_{x \to -3^+} -\frac{1}{x-5} = -\frac{1}{8}$$

  These limits do not agree but they exist. Therefore, there is a jump discontinuity.

- Consider $x = 5$. The limit from below is:

  $$\lim_{x \to 5^-} \frac{x+3}{|x-5| \cdot |x+3|} = \lim_{x \to 5^-} \frac{x+3}{-(x-5)(x+3)} = \lim_{x \to 5^-} -\frac{1}{x-5} = \lim_{x_0 \to 0^-} -\frac{1}{x_0} = \infty$$

  This is enough to say that there exists a vertical asymptote at $x = 5$.

Therefore, discontinuities exist only at $x = -3, 5$, where $x = 5$ is a vertical asymptote. □

**LA08**. Suppose $A, B \in \mathbb{R}$, $A > 0$, $B > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a function such that if $|x - y| < A$ then $|f(x) - f(y)| < B|x - y|$ for all $x, y \in \mathbb{R}$.

Prove that $f$ is continuous on $\mathbb{R}$.

*Proof.* Let $A$ and $B$ be positive reals, and $f$ be a function on the reals such that $|x-y| < A$ implies $|f(x) - f(y)| < B|x - y|$ for any $x$ and $y$.

We must show that $\lim_{n \to a} f(n) = f(a)$ for all $a$. That is, for any tolerance $\epsilon > 0$, we may find a $\delta$ such that $0 < |n - a| < \delta$ implies $|f(n) - f(a)| < \epsilon$.

Let $\epsilon > 0$ and $a$ be a real. Select $\delta = \min(\{A, \frac{\epsilon}{B}\})$.

Suppose that $0 < |n - a| < \delta$. That is, $|n - a| < A$ and $|n - a| < \frac{\epsilon}{B}$.

It also follows that $|f(n) - f(a)| < B|n - a|$. But we supposed that $|n - a| < \frac{\epsilon}{B}$, so

$$|f(n) - f(a)| < B\frac{\epsilon}{B} = \epsilon$$

This is exactly what was needed to show that $f$ is continuous for any $a$. □

**LA09**. Prove that $x^2 + x \cos x = 1$ has at least two real solutions.

*Proof.* Let $f(x) = x^2 + x \cos x$. Recall that polynomials and cosine are both continuous on $\mathbb{R}$. Therefore, their sum/product, $f$, is also continuous.

At $x = 0$, we have $f(x) = 0 + 0 = 0$.

At $x = -\pi$, we have $f(x) = \pi^2 - \pi(-1) = \pi^2 + \pi > 1$. We then have that $f(-\pi) < 1 < f(0)$. So, by the intermediate value theorem, there exists some $a \in (-\pi, 0)$ where $f(x) = 1$.

Likewise at $x = \pi$, we have $f(x) = \pi^2 + \pi(-1) = \pi^2 - \pi > 1$. We then have that $f(0) < 1 < f(\pi)$. So, by the intermediate value theorem, there exists some $b \in (0, \pi)$ where $f(x) = 1$.

We know that $a$ and $b$ are distinct because they exist on disjoint intervals.

Therefore, there must exist at least two distinct real solutions to $x^2 + x \cos x = 1$. □