Ch 5: Numerical Growth

5.1 Intro to Series

Def: Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

need not to start at 1

* This is a formal expression since we don't know what this means numerically

Ex. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (Harmonic Series)

$\sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{2}\right) = \sin(0) + \sin\left(\frac{\pi}{4}\right) + \sin(\pi) + \dots = 0 + 1 + 0 - 1 + 0 + 1 + 0 - 1 + \dots$

$\sum_{n=1}^{\infty} (-n)^n = (-1)^1 + (-2)^2 + (-3)^3 + (-4)^4 + \dots = -1 + 4 - 27 + 256 + \dots$

Def: Partial Sums

If $\sum_{n=1}^{\infty} a_n$ is a series, its sequence of partial sums, $\{S_n\}$ is defined as

$S_k = a_1 + a_2 + \dots + a_k$ (sum up to a_k)

Ex.1 For $\sum_{n=1}^{\infty} n$,
$S_1 = 1,\ S_2 = 3,\ S_3 = 6,\ S_4 = 10$ (For this series $S_k = \frac{k(k+1)}{2}$)

Def: Convergence of a Series

A series $\sum_{n=1}^{\infty} a_n$ converges to $S \in \mathbb{R}$ if $\lim_{k\to\infty} S_k = S$ and we call S the sum of the series.
If $\{S_k\}$ diverges, the series diverges.

Ex.1 $\sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{2}\right) = 0 + 1 + 0 - 1 + 0 + 1 + 0 - 1 + \dots$
has partial sums $S_0 = 0,\ S_1 = 1,\ S_2 = 1,\ S_3 = 0,\ S_4 = 0,\ S_5 = 1,\ S_6 = 1,\ S_7 = 0,\ S_8 = 0, \dots$
$\lim_{k\to\infty} S_k$ DNE, so the series diverges.

Ex.2 $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$
has partial sums $S_1 = 1 - \frac{1}{2},\ S_2 = 1 - \frac{1}{3},\ S_3 = 1 - \frac{1}{4} \dots$
So $S_k = 1 - \frac{1}{k+1}$
Since $\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = 1$, then the series converges and $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ (Telescoping Series)

The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
Assume for a contradiction that the series converges to $S \in \mathbb{R}$.

$$\begin{aligned} \text{Then } S &= \left(1 + \tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6}\right) + \left(\tfrac{1}{7} + \tfrac{1}{8}\right) + \dots \\ &> \left(\tfrac{1}{2} + \tfrac{1}{2}\right) + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{6} + \tfrac{1}{6}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8}\right) + \dots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots \\ &= S \end{aligned}$$

Hence $S > S$. Thus a contradiction, so the Harmonic Series diverges.

Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

5.2 Geometric Series

Def: Geometric Series

A geometric series is a series of the form

$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \ldots$ for some $r \in \mathbb{R}$

Case 1: $r = 1$

$\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \ldots$

$S_k = k+1$ and $\lim_{k\to\infty} S_k = \infty$, so the series diverges.

Case 2: $r = -1$

$\sum_{n=0}^{\infty} (-1)^n$

$S_k = \begin{cases} 1, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}$ and $\lim_{k\to\infty} S_k$ DNE, so the series diverges.

Case 3: $r \neq \pm 1$

Then $S_k = 1 + r + r^2 + \ldots + r^k \Rightarrow rS_k = r + r^2 + r^3 + \ldots + r^{k+1}$

So $S_k - rS_k = 1 - r^{k+1} \Rightarrow S_k = \frac{1-r^{k+1}}{1-r}$

Thus $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r} = \frac{1}{1-r} \lim_{k\to\infty} 1 - r^{k+1} = \frac{1}{1-r}$

exactly when $|r| < 1$. The limit DNE otherwise

Thm 1: Geometric Series Test

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise.
If $|r| < 1$, then

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1-r}.$$

5.4 Arithmetic of Series

Thm 3: Arithmetic for Series I

Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B_n$ and $c \in \mathbb{R}$. Then

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Thm 4: Arithmetic for Series II

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ converges for each $j \geq 1$.
2. If $\sum_{n=j}^{\infty} a_n$ converges for some j, then $\sum_{n=1}^{\infty} a_n$ converges.

Note: $\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \ldots + a_{j-1}}_{\text{finite sum}} + \sum_{n=j}^{\infty} a_n$.

Changing finite sums doesn't affect convergence.

This shows convergence only depends on the tail of the series.
We can change finitely many terms and not affect convergence.

5.2 Geometric Series (cont'd)

Ex.1 $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5}$ since $\left|-\frac{1}{4}\right| < 1$

Ex.2 $\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}}_{\text{*reindex}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2}\left(\frac{1}{1-\left(\frac{1}{2}\right)}\right) = 1$ since $\left|\frac{1}{2}\right| < 1$

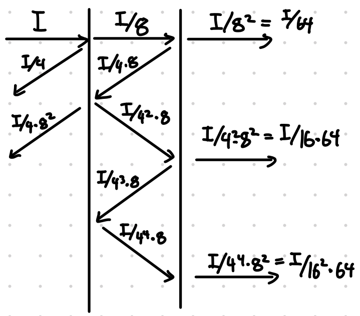
Ex.3 $\sum_{n=0}^{\infty} \frac{4^{2n}}{5^{3n+1}} = \sum_{n=0}^{\infty} \frac{(4^2)^n}{(5^3)^n \cdot 5} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{16}{125}\right)^n = \frac{1}{5}\left(\frac{1}{1-\frac{16}{125}}\right) = \frac{25}{109}$ since $\left|\frac{16}{125}\right| < 1$

Ex.4 $\sum_{n=1}^{\infty} 16 \cdot 25^n 4^{-2n-2} = \sum_{n=1}^{\infty} \frac{16 \cdot 25^n}{4^{2n} \cdot 4^2} = \sum_{n=1}^{\infty} \left(\frac{25}{16}\right)^n = \overset{\text{*reindex}}{\sum_{n=0}^{\infty} \left(\frac{25}{16}\right)^{n+1}} = \frac{25}{16} \sum_{n=0}^{\infty} \left(\frac{25}{16}\right)^n$ diverges since $\left|\frac{25}{16}\right| > 1 \Rightarrow$ original series diverges

Ex5 Express $11.62\overline{18}$ as a fraction

$$11.62\overline{18} = 11.62 + \frac{18}{10000} + \frac{18}{1000000} + \frac{18}{100000000} + \dots$$
$$= \frac{1162}{100} + \frac{18}{10^4} + \frac{18}{10^6} + \frac{18}{10^8} + \dots$$
$$= \frac{1162}{100} + \frac{18}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\right)$$
$$= \frac{1162}{100} + \frac{18}{10^4}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \dots\right)$$
$$= \frac{1162}{100} + \frac{18}{10^4}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n$$
$$= \frac{1162}{100} + \frac{18}{10^4}\left(\frac{1}{1-1/100}\right)$$
$$= \frac{3196}{275}$$

Ex6 A laser beam is fired at a planet with an innershield and outershied. Each shield reflects $\frac{1}{4}$ absorbs $\frac{5}{6}$, and transmits $\frac{1}{8}$ of the beam. If the beam's intensity is I, what fraction passes through both shields



$$\frac{I}{64} + \frac{I}{16\cdot 64} + \frac{I}{16^2\cdot 64} + \frac{I}{16^3\cdot 64} + \dots$$
$$= \frac{I}{64}\left(1 + \frac{1}{16} + \frac{1}{16^2} + \frac{1}{16^3} + \dots\right)$$
$$= \frac{I}{64}\sum_{n=0}^{\infty}\left(\frac{1}{16}\right)^n$$
$$= \frac{I}{64}\left(\frac{1}{1-1/16}\right)$$
$$= \frac{I}{64}\cdot\frac{16}{15}$$
$$= \frac{I}{60} \quad \text{since } \left|\frac{1}{16}\right| < 1$$

5.3 Divergence Test

We can focus on determining if a series converges or diverges generally without finding the sum.

Thm

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n = S$ for some $S \in \mathbb{R}$.

Proof

Let $\{S_k\}$ be the sequence of partial sums, so

$S_n = a_1 + a_2 + \dots + a_k + \dots + a_n$, $S_{n-1} = a_1 + a_2 + \dots + a_k + \dots + a_{n-1}$

Note: $\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{n-1}$

Since $a_n = S_n - S_{n-1}$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = S - S = 0$ ■

Thm 2: Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

Contrapositive: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Div test only says if a series diverges, never converges

$\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

$\lim_{n\to\infty} a_n = 0 \Rightarrow$ Div test fails (inconclusive)

Note: Div test can be used to spot a divergent series before using a complicated test, so try it 1st

Ex.1 $\sum_{n=1}^{\infty} \frac{n}{n+10^{10}}$ Since $\lim_{n\to\infty} \frac{n}{n+10^{10}} = 1 \neq 0$, the series diverges by Div test.

Ex.2 $\sum_{n=1}^{\infty} (-1)^n$ Since $\lim_{n\to\infty} (-1)^n$ DNE, the series diverges by Div test

Ex.3 $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ Since $\lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty$, the series diverges by Div test

Ex.4 $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ Since $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$, so Div test is inconclusive

5.5 & 5.6 Tests for Positive Series

Def: Positive Series

A series $\sum_{n=1}^{\infty} a_n$ is positive if $a_n \geq 0 \ \forall n \in \mathbb{N}$

The Integral Test

Thm 8: Integral Test

Let f be a function that is
1. continuous on $[0,\infty)$
2. positive on $[1,\infty)$
3. decreasing on $[1,\infty)$

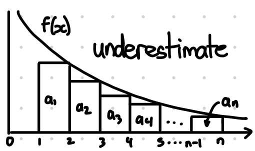
and let $a_n = f(n)$. Then

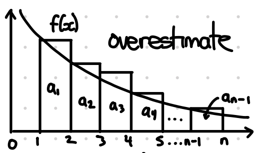
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

We only need the 3 conditions to hold on $[m,\infty)$ for some $m \geq 1$ (eventually) because we only care if the tail of a_n converges.

Proof

Let f be cts, pos, decr on $[1,\infty)$. Let $y=f(x)$ with $f(n)=a_n$.





Since f is cts, pos, decr, then

$$\sum_{k=2}^{n} \leq \int_1^n f(x)\,dx \quad \text{and} \quad \sum_{k=1}^{n} a_n \geq \int_1^n f(x)\,dx$$

($\Leftarrow$) Suppose $\int_1^{\infty} f(x)$ converges.

We know $\sum_{k=2}^{n} \leq \int_1^n f(x)\,dx \leq \int_1^{\infty} f(x)\,dx$ since $f(x)$ is pos.

Since $\int_1^{\infty} f(x)\,dx$ converges, $S_n = a_1 + \sum_{k=2}^{n} a_k \leq a_1 + \int_1^{\infty} f(x)\,dx = M$ where M is a constant.

Since $a_n = f(n) \geq 0$, we have $0 \leq S_n \leq M \ \forall n \in \mathbb{N}$ and $S_{n+1} = S_n + a_{n+1} \geq S_n$

$\therefore \{S_n\}$ is a bounded monotonic sequence and converges by MCT. Thus $\sum_{n=1}^{\infty} a_n$ converges.

($\Rightarrow$) Suppose $\int_1^{\infty} f(x)\,dx$ diverges from above, so $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$

We know $\int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} a_n = S_{n-1}$, so $\lim_{n\to\infty} S_{n-1} = \infty$

$\therefore \sum_{n=1}^{\infty} a_n$ diverges ■

Ex.1 $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Consider $f(x)=\frac{1}{x^2}$, which is cts, pos, and decr on $[1,\infty)$ so the Integral Test applies.

$\int_1^{\infty} \frac{1}{x^2}dx$ converges (p-test $p=2>1$), so $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

Ex.2 $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{LHR\ (\frac{\infty}{\infty})}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ so the Div Test fails.

Consider $f(x)=\frac{\ln(x)}{x}$ where f is cts and pos on $[1,\infty)$.

For decr, $f'(x)=\frac{1-\ln x}{x}\le 0 \Rightarrow 1-\ln x\le 0 \Rightarrow 1\le \ln x \Rightarrow x\ge e$.

Thus f is decr on $[e,\infty)$ and the Integral Test applies

$\int_1^{\infty} \frac{\ln(x)}{x}dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x}dx = \lim_{b\to\infty} \frac{(\ln(x))^2}{2}\Big|_1^b = \infty$, so $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ diverges.

Thm 9: p-Series Test

The series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p>1$ and diverges for $p\le 1$

Ex. $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (p-series, $p=\frac{3}{2}>1$)

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series, $p=\frac{1}{2}\le 1$)

Note that we can't use the Integral Test to compute the sum of a convergent series

For example, $\int_1^{\infty}\frac{1}{x^2}dx=1$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

But we can approximate sums with the Integral Test.

Def: Remainder

For a convergent series $\sum_{n=1}^{\infty} a_n = S\in\mathbb{R}$, the remainder is the error when using S_n to approximate S is the error when using S_n to approximate S is given by

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \dots$$

Thm 8: Integral Test and Estimation of Sums and Errors

If $a_n = f(x)$ where $f(x)$ is cts, pos, and decr on $[1,\infty)$, then

$$\int_{n+1}^{\infty} f(x)dx \le \underset{\uparrow \atop S-S_n}{R_n} \le \int_n^{\infty} f(x)dx$$ (from the proof of the Integral Test)

We get the upper bound on the remainder.

We can improve our estimate, rather than just use S_n,

$$S_n + \int_{n+1}^{\infty} f(x)dx \le S \le S_n + \int_n^{\infty} f(x)dx$$

Take the midpoint to approximate S with the error at most half of the width of the interval.

Ex.1 Find an upper bound on the error if we use S_{10} to approximate $\sum_{n=1}^{\infty}\frac{1}{n^3}$

$$R_{10} \le \int_{10}^{\infty}\frac{1}{x^3}dx = \lim_{b\to\infty}\int_{10}^{b}\frac{1}{x^3}dx = \lim_{b\to\infty} -\frac{1}{2x^2}\Big|_{10}^{b} = \lim_{b\to\infty} -\frac{1}{2b^2} - \left(-\frac{1}{2(10)^2}\right) = \frac{1}{200} = 0.005$$

$\Rightarrow$ the error is at most 0.005

Ex.2 How many terms are needed to approximate $\sum_{n=1}^{\infty}\frac{1}{n^3}$ with an error of at most 0.00005

$$R_n \le \int_{n}^{\infty}\frac{1}{x^3}dx = \frac{1}{2n^2}$$

We need $\frac{1}{2n^2} \le 0.0005$

$\Rightarrow 2n^2 \ge 2000$

$\Rightarrow n^2 \ge 1000$

$\Rightarrow n \ge \sqrt{1000} \approx 31.23$

$\Rightarrow n \ge 32$ for the desired accuracy

Ex.3 Considering $\sum_{n=1}^{\infty}\frac{1}{n^3}$, we know using S_{10} to approximate S

$$S_{10} + \int_{11}^{\infty}\frac{1}{x^3}dx \le S \le S_{10} + \int_{10}^{\infty}\frac{1}{x^3}dx$$

$\Rightarrow 1.201664 \le S \le 1.202532\ldots$

*better than Ex.1

$\therefore S \approx 1.202098\ldots$ and the error is actually 0.0005 (round up)

The Comparison Test

Thm 6: Comparison Test

Assume $0 \le a_n \le b_n$ for $n \in \mathbb{N}$ (or eventually)

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof of (2):

Let $\{S_n^a\}$ and $\{S_n^b\}$ be the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively

If $\sum_{n=1}^{\infty} a_n$ diverges, then since $a_n \ge 0\ \forall n$, $\lim_{n\to\infty} S_n^a = \infty$

But since $a_n \le b_n\ \forall n$, $S_n^a \le S_n^b\ \forall n$, we have $\lim_{n\to\infty} S_n^b = \infty$

$\because b_n$ diverges

$\therefore$ (1) is the contrapositive of (2) and so is true. ■

Ex.1 $\sum_{n=1}^{\infty}\frac{n^2}{n^4+16}$

$$0 \le \frac{n^2}{n^4+16} \le \frac{n^2}{n^4} = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (p-series, $p=2>1$), the given series converges by comparison.

Ex.2 $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$

For $n \ge e$, $\frac{\ln(n)}{n} \ge \frac{1}{n} \ge 0$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (Harmonic series), the given series diverges by comparison.

Ex.3 $\sum_{n=1}^{\infty}\frac{n^2+n}{n^4-8}$

For $n \ge 2$, $\frac{n^2+n}{n^4-8} \ge \frac{n^2+n}{n^4} \ge \frac{n^2}{n^4} = \frac{1}{n^2} \ge 0$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series $p=2>1$), the comparison test is inconclusive.

But this series "looks like" $\sum_{n=1}^{\infty}\frac{1}{n^2}$, so we expect it to converge.

The Limit Comparison Test

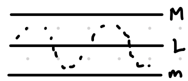
Thm 7. Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n > 0$ for $n \in \mathbb{N}$ (or eventually) and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ then

1. If $L \in (0,\infty)$, then Σa_n converges $\Leftrightarrow$ Σb_n converges
2. If $L = 0$ and Σb_n converges, then Σa_n converges.
   If $L = 0$ and Σa_n diverges, then Σb_n diverges.
3. If $L = \infty$ and Σa_n converges, then Σb_n converges.
   If $L = \infty$ and Σb_n diverges, then Σa_n diverges.

Proof (1):

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty)$. Then $\exists\, m, M \in (0,\infty)$ so that $m \leq \frac{a_n}{b_n} \leq M$ for n sufficient large.



We get $mb_n \leq a_n \leq Mb_n$ for n sufficient large.
By Comparison Test Σa_n converge/diverge iff Σb_n converge/diverge. ■

Proof Idea for (3):

$0 \leq m \leq \frac{a_n}{b_n} \Rightarrow mb_n \leq a_n$

Ex.1 $\sum_{n=1}^{\infty} \frac{n^2+n}{n^4-8}$

Apply LCT with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{n^4-8}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3}{n^4-8} = 1 \in (0,\infty)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$), the given series converges by LCT

Ex.2 $\sum_{n=1}^{\infty} \frac{4^n+1}{3^n+n}$

Apply LCT with $\sum_{n=1}^{\infty} \frac{4^n}{3^n}$

$$\lim_{n\to\infty} \frac{\frac{4^n+1}{3^n+n}}{\frac{4^n}{3^n}} = \lim_{n\to\infty} \frac{12^n+3^n}{12^n+n4^n} = \lim_{n\to\infty} \frac{12^n(1+\frac{1}{4^n})}{12^n(1+\frac{n}{3^n})} = 1 \in (0,\infty)$$

Since $\sum_{n=1}^{\infty} \frac{4^n}{3^n}$ diverges (geo with $|r| = |\frac{4}{3}| > 1$), the given series converges by LCT

Note: Div test could work

Ex.3 $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Use LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln(n) = \infty$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic series), the given series diverges.

5.7 Alternating Series

Def: Alternating Series

A series is alternating if terms are alternating positive and negative.
Series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

$$\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \ldots$$

provided that $a_n > 0\ \forall n$.

Ex.1 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Thm 10: Alternating Series Test (AST)

Let $a_n > 0\ \forall n$ and consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. If

1. $\{a_n\}$ is (eventually) decreasing: $a_n \geq a_{n+1}$
2. $\lim_{n\to\infty} a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof:

Suppose $\{a_n\}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$.
We prove $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit.
$\{a_n\}$ decreasing $\Rightarrow a_j - a_{j+1} > 0 \ \forall j$. Hence

For even partial sums:

$S_2 = a_1 - a_2 > 0$

$S_4 = (a_1-a_2)+(a_3-a_4) = S_2 + (a_3 - a_4) > S_2$

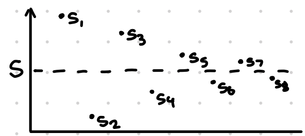
$S_6 = S_4 + (a_5 - a_6) > S_4$

Thus $0 < S_2 < S_4 < \ldots < S_{2n} < \ldots$, so $\{S_{2n}\}$ is increasing.
Now $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n+1}) - a_{2n} \Rightarrow S_{2n} \le a_1$
Since $\{S_{2n}\}$ is increasing and bounded above,
it converges by MCT. Say $\lim_{n\to\infty} S_{2n} = S$.

For odd partial sums:

Since $S_{2n+1} = S_{2n} + a_{2n+1}$ and $\lim_{n\to\infty} a_n = 0$.
We have $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + a_{2n+1}) = S + 0 = S$.

So $\lim_{n\to\infty} S_{2n} = S = \lim_{n\to\infty} S_{2n+1}$ so $\lim_{n\to\infty} S_n = S$.
Thus the series converges. ∎



Ex.1 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (Alternating Harmonic Series)

For $n \ge 1$, $\frac{1}{n} > 0$, $\frac{1}{n+1} < \frac{1}{n}$, and since $\lim_{n\to\infty} \frac{1}{n} = 0$,
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST (to $\ln 2$).

Ex.2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{ne^n}$

For $n \ge 1$, $\frac{1}{ne^n} > 0$, $\frac{1}{(n+1)e^{n+1}} < \frac{1}{ne^n}$, and since $\lim_{n\to\infty} \frac{1}{ne^n} = 0$,
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{ne^n}$ converges.

Thm 10: Alternating Series Test (AST) and Estimation of Sums and Errors

Suppose $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to S.
From the proof, the odd/even partial sums approach the actual sum from above/below.
$\Rightarrow S$ lies between any 2 consective partial sums
$\Rightarrow |R_n| = |S - S_n| \le |S_{n+1} - S_n| = |\pm a_{n+1}| = a_{n+1}$
The error is at most a_{n+1}. That is

$|R_n| = |S - S_n| \le a_{n+1}$

Ex.1 Find an upper bound on the error if we use S_6 to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$

$|R_6| \le a_7 = \frac{1}{e^7} \approx 0.00092$ (rounded up)

Ex.2 How many terms are needed to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!e^n}$ with an error of at most 0.000005

$|R_n| \le a_{n+1} = \frac{1}{(n+1)e^{n+1}} \Rightarrow \frac{1}{(n+1)e^{n+1}} \le 0.000005$

$n=4$: $\frac{1}{5!e^5} \approx 0.000056\ldots$ (nope)

$n=5$: $\frac{1}{6!e^6} \approx 0.00000344\ldots$ (yup)

Thus 5 terms are needed.

If the 1st term is positive, then
- odd partial sums are overestimates
- even partial sums are underestimates

If the 1st term is negative, then
- odd partial sums are underestimates
- even partial sums are overestimates

Ex.1 Is S_{4032} an underestimate or overestimate of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{ne^n}$?
1st term is positive so S_{4032} is an underestimate

5.8 Absolute vs Conditional Convergence

Def: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Note: $a_n \geq 0\ \forall n \Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$

Ex.1 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent

since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges (p-series $p=3>1$)

Ex.2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent

since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

which is divergent (Harmonic Series)

Def: Conditionally Convergent

A series is conditionally convergent if it is convergent but not absolutely convergent.

Thm 11: Absolute Convergence Theorem (ACT)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is convergent

(but not to the same value unless $a_n \geq 0\ \forall n$)

__Proof__

Since $0 \leq a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n| \Rightarrow a_n + |a_n| \leq 2|a_n|$

For each n, we have $\sum_{n=1}^{\infty} (a_n + |a_n|) \leq \sum_{n=1}^{\infty} 2|a_n|$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} 2|a_n|$.

So $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by Comparison.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$

Since both series on the right converges and $\sum_{n=1}^{\infty} a_n$ can be expressed as the difference of 2 convergent series
$\therefore$ it also converges ■

This allows us to use tests for positive series on any series by taking the absolute value.

Ex.1 Is $\sum_{n=1}^{\infty} \frac{\cos(n)+\sin(n)}{2n^2}$ convergent? (+, +, −, −, +, +, +)

We check for convergence by considering $\sum_{n=1}^{\infty} \left| \frac{\cos(n)+\sin(n)}{2n^2} \right|$

$$0 \leq \left| \frac{\cos(n)+\sin(n)}{2n^2} \right| = \frac{|\cos(n)+\sin(n)|}{2n^2} \underset{\text{by triangle inequality}}{\leq} \frac{|\cos(n)|+|\sin(n)|}{2n^2} < \frac{2}{2n^2} = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2>1$), $\sum_{n=1}^{\infty} \left| \frac{\cos(n)+\sin(n)}{2n^2} \right|$ converges by Comparison

By ACT, $\sum_{n=1}^{\infty} \frac{\cos(n)+\sin(n)}{2n^2}$ converges.

To determine absolute/conditional convergence or divergence.

1. Divergence Test
2. Check for Absolute convergence using Tests for positive series
3. Check for Conditional convergence using AST (if possible)

Ex.1 $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n^2+1}}{n^2}$ converge abs/cond or diverges

1. $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n^2} = \lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n^2}}}{n^2} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{n} = 0$

   $\therefore$ Div test is inconclusive

2. Consider $\sum_{n=1}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n^2+1}}{n^2}$

   Use LCT with $\sum_{n=1}^{\infty}\frac{1}{n}$

   $\lim_{n\to\infty}\frac{\frac{\sqrt{n^2+1}}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty}\frac{n\sqrt{1+\frac{1}{n^2}}}{n} = 1 \in (0,\infty)$

   Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (Harmonic Series), $\sum_{n=1}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2}\right|$ diverges

   $\therefore \sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2}$ doesn't converge absolutely

3. We know $\frac{\sqrt{n^2+1}}{n^2} \geq 0 \ \forall n$ and $\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{n^2} = 0$ by 1.

   Let $f(x) = \frac{\sqrt{x^2+1}}{x^2} \Rightarrow f'(x) = -\frac{(x^2+2)}{x^3\sqrt{x^2+1}} < 0$ for $x>0 \Rightarrow \left\{\frac{\sqrt{n^2+1}}{n^2}\right\}_{n=1}^{\infty}$ is decreasing

   So by AST, $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2}$ converges

   $\therefore$ Finally $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2}$ converges conditionally

Thm 12: Rearrangement Theorem

1. If $\sum_{n=1}^{\infty} a_n$ absolutely converges to $S\in\mathbb{R}$, then rearranging the terms will still converge to S.
2. If $\sum_{n=1}^{\infty} a_n$ conditionally converges to $S\in\mathbb{R}$, then rearranging the terms will change the sum to $\alpha\in\mathbb{R}$ or $\pm\infty$.

5.9 Ratio Test

Thm 13: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and assume that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, $L\in[0,\infty)$ or $L=\infty$.

1. If $L<1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L>1$ or $L=\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L=1$, then the Ratio test is inconclusive.

<u>Proof</u>:

1. If $0\leq L<1$, $\exists r\in\mathbb{R} \ni L<r<1$.

   Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<r$, $\exists N\in\mathbb{N} \ni \forall n\geq N$, $\left|\frac{a_{n+1}}{a_n}\right|<r \Rightarrow |a_{n+1}|<r|a_n|$.

   Thus $|a_{n+1}|<r|a_n|$, $|a_{n+2}|<r|a_{n+1}|<r^2|a_n|$ so $|a_{N+k}|<r^k|a_N|$ (*)

   Since $|r|=r<1$, $\sum_{n=1}^{\infty}|a_N|r^n$ is a convergent geometric series.

   By Comparison and using (*), $\sum_{n=N+1}^{\infty}|a_n|$ converges.

   Why? $\sum_{n=1}^{\infty}|a_n| = |a_{N+1}|+|a_{N+2}|+|a_{N+3}|+\ldots < r|a_N|+r^2|a_N|+r^3|a_N|+\ldots$ (by (*)) $=\sum_{n=1}^{\infty}|a_N|r^n$.

   Thus $\sum_{n=1}^{\infty} a_n$ converges absolutely.

2. If $L>1$ or $L=\infty$, then $\exists N\in\mathbb{N} \ni \forall n\geq N$, $\left|\frac{a_{n+1}}{a_n}\right|>1 \Rightarrow |a_{n+1}|>|a_n|$

   Thus $\{|a_n|\}_{n=1}^{\infty}$ will eventually increase, so $\lim_{n\to\infty} a_n \neq 0$

   So the series diverges by Div Test.

3. Consider $\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$. For both series, $L=1$, but one diverges while the other converges.

Ex.1 $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

Using Ratio test,

$$\lim_{n\to\infty}\left|\frac{5^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n}\right| = \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1.$$

$\therefore \sum_{n=1}^{\infty}\frac{5^n}{n!}$ converges absolutely.

Ex.2 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n 2^n}{5^n}$

Using Ratio test,

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)2^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(-1)^{n+1}n2^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)2(n+1)}{5n}\right| = \frac{2}{5}\lim_{n\to\infty}\frac{n+1}{n} = \frac{2}{5} < 1.$$

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n2^n}{5^n}$ converges absolutely.

Ex.3 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Using Ratio test,

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n}{n^n}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{|n+1|n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

$\therefore \sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges.

Ex.4 $\sum_{n=1}^{\infty} \frac{n^3+4}{n^5-1}$

Using Ratio test,

$$\lim_{n\to\infty}\left|\frac{(n+1)^3+4}{(n+1)^5-1}\cdot\frac{n^5-1}{n^3+4}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^3+4}{n^3+4}\cdot\frac{n^5-1}{(n+1)^5-1}\right) = 1\cdot 1 = 1$$

$\therefore$ inconclusive $\to$ try LCT

5.10 Root Test

Thm 15: Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and assume $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$, $L\in[0,\infty)$ or $L=\infty$.

1. If $L<1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L>1$ or $L=\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L=1$, then the Root test is inconclusive.

Ex.1 $\sum_{n=1}^{\infty}\left(\frac{\ln(n+1)}{n^3}\right)^n$

Using Root test,

$$\lim_{n\to\infty}\left|\left(\frac{\ln(n+1)}{n^3}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{\ln(n+1)}{n^3} \overset{\text{L'HR}}{=} \lim_{n\to\infty}\frac{\frac{1}{n+1}}{3n^2} = \lim_{n\to\infty}\frac{1}{3n^2(n+1)} = 0$$

$\therefore$ series converges

Note. If the Root test $(L=1)$, the Ratio test will fail.

Relative size of common functions:

$$(\ln n)^p \ll n^p \ll x^n \ll n! \ll n^n, \quad |x|>1$$

$$\frac{1}{n^n} \ll \frac{1}{n!} \ll \frac{1}{x^n} \ll \frac{1}{n^p} \ll \frac{1}{(\ln n)^p}, \quad |x|>1$$

Thm 14: Polynomial vs Factorial Growth

For $x\in\mathbb{R}$, $\lim_{n\to\infty}\frac{x^n}{n!} = 0$.

Recap of Series Test

Absolute Convergence or Divergence:

- Sums of Geometric and Telescoping Series
  - ↳ try to spot these series
  - ↳ if it asks to find the sum, it's likely one of these
- Divergence Test (any series)
  - ↳ try this 1st, unless there's a factorial
- Integral Test (positive series)
  - ↳ last resort when all else fails
  - ↳ don't forget: continuous, positive, decreasing
- Comparison Test (positive series)
  - ↳ last resort when all else fails
- LCT (positive series)
  - ↳ series of the form "$\frac{\text{powers of } n}{\text{powers of } n}$"
  - ↳ "almost" geometric series
  - ↳ don't forget: $L=0$ or $L=\infty$ are more complicated
- Ratio Test (any series)
  - ↳ factorials
  - ↳ "almost" geometric series
  - ↳ $L=1$ gives no info
- Root Test (any series)
  - ↳ when all terms have a power of n

Conditional Convergence:

- AST (alternating series)
  - ↳ look for $(-1)^n$ or $(-1)^{n+1}$