Ch 6: Power Series

6.1 Intro to Power Series

Def: Power Series

A power series is a series of the form

$\sum_{n=0}^{\infty} a_n x^n = a_1 x + a_2 x_2^2 + \dots$ (centered at 0)

$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$ (centered at a)

where $a_i \in \mathbb{R}$ $\forall i$.

The domain of a power series is the collection of all $x \in \mathbb{R}$ for which the power series converges.

Note: $\sum_{n=0}^{\infty} a_n (x-a)^n$ will converge to a_0 at $x=a$ (the center) so the domain is never empty

Note: For $\sum_{n=0}^{\infty} a_n (x-a)^n$, we follow
1. For $n=0$, $a_0(x-a)^0 = a_0$ $(0^0 = 1)$
2. If $a_0 = a_1 = \dots = a_k = 0$, then $\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=k+1}^{\infty} a_n (x-a)^n$ (discard terms with 0)

Ex.1 Find the domain of $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.
Use Ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 > 1$$

This holds $\forall x \in \mathbb{R}$ so the domain is $\mathbb{R}$.

Ex.2 Find the domain of $\sum_{n=0}^{\infty} (x-3)^n$.
Use Root test.

$$\lim_{n\to\infty} \left(|(x-3)^n| \right)^{\frac{1}{n}} = \lim_{n\to\infty} |x-3| = |x-3|$$

We need $|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$.
The Root test fails when $|x-3| = 1$, when $x = 2, 4$.

Test these separately:
For $x=2$, $\sum_{n=0}^{\infty} (-1)^n$
For $x=4$, $\sum_{n=0}^{\infty} 1^n$
both are geometric series with $|r|=1$, so both diverge
The domain is $(2,4)$.

Thm 1: Fundamental Convergence Theorem for Power Series

For $\sum_{n=0}^{\infty} a_n (x-a)^n$, one of the following must hold.
1. The series converges only when $x=a$, $[a,a]$.
2. The series converges $\forall x \in \mathbb{R}$, $(-\infty, \infty)$.
3. $\exists R \in \mathbb{R} \ni$ the series converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$, and may converge/diverge for $|x-a| = R$.

Proof wlog:
Assume $a=0$ so our power series is $\sum_{n=0}^{\infty} a_n x^n$. Thus centered at 0. So it converges for $x=0$.
Assume our power series converges for some nonzero $x = x_0 \in \mathbb{R}$.
We show that if $|x_1| < |x_0|$, then $\sum_{n=1}^{\infty} |a_n x_1^n|$ converges.
Since $\sum_{n=1}^{\infty} a_n x_0^n$ converges, $\lim_{n\to\infty} |a_n x_0^n| = 0$ by the Div test.

Thus $|a_n x_0^n| < 1$ eventually.

Thus $|a_n x_1^n| = |a_n x_0^n| \cdot \left| \frac{x_1^n}{x_0^n} \right| \le \left| \frac{x_1^n}{x_0^n} \right|$ eventually

Since $\sum_{n=0}^{\infty} \left| \frac{x_1}{x_0} \right|^n$ converges (geo series $|r| = \left|\frac{x_1}{x_0}\right| < 1$), $\sum_{n=0}^{\infty} |a_n x_1^n|$ converges by comparison. ■

Note: The domain is always a single interval.

R is the radius of convergence.
1. $\Rightarrow R=0.$
2. $\Rightarrow R=\infty.$
3. $\Rightarrow R\in(0,\infty)$

The Interval of Convergence is the interval where it converges.
1. $\Rightarrow I=\{a\}$
2. $\Rightarrow I=(-\infty,\infty)=\mathbb{R}$
3. $\Rightarrow I=(a-R,a+R),\ I=[a-R,a+R],\ I=(a-R,a+R],\ I=[a-R,a+R)$

To find the radius, use the Ratio test (Ex.6 in 6.1 shows this can fail).

Ex.1 $\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n}}$

Use Ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{2^n(x-3)^n}\right|=\lim_{n\to\infty}\frac{2|x-3|\sqrt{n}}{\sqrt{n+1}}=2|x-3|$$

We need $2|x-3|<1\Rightarrow|x-3|<\frac{1}{2}$. So $R=\frac{1}{2}$.

The series converges for $x\in(3-\frac{1}{2},3+\frac{1}{2})=(\frac{5}{2},\frac{7}{2})$

Check endpoints:

$x=\frac{5}{2}$: $\sum_{n=1}^{\infty}\frac{2^n(\frac{5}{2}-3)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{2^n(-\frac{1}{2})^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Since $\frac{1}{\sqrt{n}}>0$, $\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$, $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by AST.

$x=\frac{7}{2}$: $\sum_{n=1}^{\infty}\frac{2^n(\frac{7}{2}-3)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{2^n(\frac{1}{2})^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series, $p=\frac{1}{2}\le 1$)

$\therefore I=[\frac{5}{2},\frac{7}{2})$

Ex.2 $\sum_{n=0}^{\infty}n!\,x^n$

Use Ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right|=\lim_{n\to\infty}(n+1)|x|=\begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\neq 0\end{cases}$$

Diverges unless $x=0$. So $R=0$, $I=\{0\}$.

Ex.3 $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{3^n\ln(n)}$

Use Ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}}{3^{n+1}\ln(n+1)}\cdot\frac{3^n\ln(n)}{(-1)^{n+1}x^{n+1}}\right|=\lim_{n\to\infty}\frac{|x|\ln(n)}{3\ln(n+1)}=\frac{1}{3}|x|$$

We need $\frac{1}{3}|x|<1\Rightarrow|x|<3$. So $R=3$ and open interval is $(-3,3)$.

Check endpoints:

$x=-3$: $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(-3)^{n+1}}{3^n\ln(n)}=3\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$

Note: $\frac{1}{\ln(n)}\ge\frac{1}{n}$ for $n\ge 2$.

Since $\frac{1}{n}$ diverges (Harmonic Series), $\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ diverges by comparison.

$x=3$ $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(3)^{n+1}}{3^n\ln(n)}=3\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln(n)}$

Since $\frac{1}{\ln(n)}>0$, $\frac{1}{\ln(n+1)}<\frac{1}{\ln(n)}$, $\lim_{n\to\infty}\frac{1}{\ln(n)}=0$, so $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln(n)}$ converges by AST.

$\therefore I=(3,-3]$

6.2 Representing Functions as Power Series

A power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a function whose domain is its interval of convergence.

We know one such function: Geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1 \quad (R=1,\ I=(-1,1))$$

We don't need to check endpoints for geometric series.

Thm 4: Abel's Theorem

If $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I, then f is continuous on I.

Thm:

We can build power series for new functions using given power series.

$f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius R_f and interval of convergence I_f.
$g(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius R_g and interval of convergence I_g.
(centered at the same a)

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-a)^n$
   If $R_f \neq R_g$, then $R=\min\{R_f, R_g\}$ and $I=I_f \cap I_g$
   If $R_f = R_g$, then $R \geq R_f = R_g$.

2. $(x-a)^k f(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+k}$
   $R=R_f$ and $I=I_f$.

3. If $c \in \mathbb{R}$, $c \neq 0$, $a=0$, $(f(x)=\sum_{n=0}^{\infty} a_n x^n)$, then $f(cx^k)=\sum_{n=0}^{\infty} a_n c^n x^{kn}$.
   If $R_f < \infty$, then $|cx^k| < R_f \Rightarrow |x| < \sqrt[k]{\frac{R_f}{|c|}}$. So $R=\sqrt[k]{\frac{R_f}{|c|}}$.
   If $R_f = \infty$, then $R=\infty$.
   The interval is $I=\{x \in \mathbb{R} \mid cx^k \in I_f\}$.

Ex.1 $f(x)=\frac{1}{5-x}$ centered at $x=0$.

$$\frac{1}{5-x} = \frac{1}{5}\left(\frac{1}{1-\frac{x}{5}}\right) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

for $\left|\frac{x}{5}\right|<1 \Rightarrow |x|<5$ So $R=5$, $I=(-5,5)$.

Ex.2 $f(x)=\frac{x^3}{x+2}$ centered at $x=0$.

$$\frac{x^3}{x+2} = \frac{x^3}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{x^3}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{n+3}}{2^{n+1}}\right)$$

for $\left|-\frac{x}{2}\right|<1 \Rightarrow |x|<2$. So $R=2$, $I=(-2,2)$.

Ex.3 $f(x)=\frac{1}{2-x^2}$ centered at $x=0$.

$$\frac{1}{2-x^2} = \frac{1}{2}\left(\frac{1}{1-\frac{x^2}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{n+1}}$$

for $\left|\frac{x^2}{2}\right|<1 \Rightarrow |x|<\sqrt{2}$. So $R=\sqrt{2}$, $I=(-\sqrt{2},\sqrt{2})$.

Ex.4 $f(x)=\frac{2}{x}$ centered at $x=7$.

$$\frac{2}{x} = \frac{2}{7+(x-7)} = \frac{2}{7}\left(\frac{1}{1+\frac{x-7}{7}}\right) = \frac{2}{7}\left(\frac{1}{1-\left(\frac{-(x-7)}{7}\right)}\right) = \frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{-(x-7)}{7}\right)^n = 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n(x-7)^n}{7^{n+1}}\right)$$

for $\left|\frac{-(x-7)}{7}\right|<1 \Rightarrow |x-7|<7$. So $R=7$, $I=(0,14)$.

6.3 & 6.4 Differention and Integration of Power Series

For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, we can differentiate or integrate term-by-term.

Thm:

If $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R>0$, then $f(x)$ is differentiable and integrable on $(a-R, a+R)$

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$

change index since if $n=0$, the term is 0

2. $\int f(x) = \sum_{n=0}^{\infty}\left(\frac{a_n(x-a)^{n+1}}{n+1}\right) + C$

both have radius of convergence R

The radius doesn't change, but the interval may change. We need to check the endpoints if we differentiate/integrate.

Ex.1 Find a power series for $\ln|1+x|$ about $x=0$.

We know $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$, so $R=1$.

We get $\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$ $(R=1)$.

Integrate: $\ln|1+x|=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}+C$, $R=1$.

Find C by subbing in $x=0$.
(since we want a series for $\ln|x+1|$ explicitly)

$\ln|1+0|=\sum_{n=0}^{\infty}\frac{(-1)^n 0^{n+1}}{n+1}+C \Rightarrow C=0$

So $\ln|1+x|=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$, $R=1$ and open interval is $(-1,1)$.

Check endpoints:

$x=1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ converges by AST

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{n+1}}{n+1}=\sum_{n=0}^{\infty}\frac{-1}{n+1}$ diverges (Harmonic Series)

$\therefore I=(-1,1]$

Ex.2 Find a power series for $f(x)=\frac{1}{(1-x)^3}$ about $x=0$.

We know $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$, so $R=1$.

Differentiate: $\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}$ $(R=1)$

Differentiate again: $\frac{2}{(1-x)^3}=\sum_{n=2}^{\infty}n(n-1)x^{n-2}$ $(R=1)$

So $\frac{1}{(1-x)^3}=\frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2}$, $R=1$ and open interal is $(-1,1)$.

Check endpoints:

$x=1$: $\frac{1}{2}\sum_{n=2}^{\infty}n(n-1)$

$x=-1$: $\frac{1}{2}\sum_{n=2}^{\infty}n(n-1)(-1)^{n-2}$

both diverge
Div test

$\therefore I=(-1,1)$.

Ex.3 Find a power series for $f(x)=\arctan(x)$ about $x=0$.

We will first find a series for $\frac{1}{1+x^2}$.

$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $|-x^2|<1 \Rightarrow |x|<1$ $(R=1)$

So $\arctan(x)=\int\frac{1}{1+x^2}dx=\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{2n+1}\right)+C$.

Sub in $x=0$ to get C: $\arctan(0)=0+C \Rightarrow C=0$.

So $\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$, $R=1$ and open interval is $(-1,1)$.

Check endpoints:

$x=1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ converges by AST.

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges by AST.

$\therefore I=[-1,1]$

Ex.4 Evalute $\int\frac{1}{2-x^5}dx$ as a power series about $x=0$.

First, find a series for $\frac{1}{2-x^5}$.

$\frac{1}{2-x^5}=\frac{1}{2}\left(\frac{1}{1-\frac{x^5}{2}}\right)=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^5}{2}\right)^n=\sum_{n=0}^{\infty}\frac{x^{5n}}{2^{n+1}}$ for $\left|\frac{x^5}{2}\right|<1 \Rightarrow |x|<2^{\frac{1}{5}}$ $(R=2^{\frac{1}{5}})$

Integrate: $\int\frac{1}{2-x^5}dx=\int\sum_{n=0}^{\infty}\frac{x^{5n}}{2^{n+1}}dx=\sum_{n=0}^{\infty}\left(\frac{x^{5n+1}}{2^{n+1}(5n+1)}\right)+C$,

(won't find C since we are evaluating an indefinite integral)

$R=2^{\frac{1}{5}}$ and open interval is $(-2^{\frac{1}{5}},2^{\frac{1}{5}})$.

Check endpoints.

$x=2^{\frac{1}{5}}$: $\sum_{n=0}^{\infty}\left(\frac{(2^{\frac{1}{5}})^{5n+1}}{2^{n+1}(5n+1)}\right)=\sum_{n=0}^{\infty}\frac{2^{\frac{1}{5}}}{2}\cdot\frac{1}{5n+1}$ diverges by LCT (with $\frac{1}{n}$)

$x=-2^{\frac{1}{5}}$: $\sum_{n=0}^{\infty}\left(\frac{(-2^{\frac{1}{5}})^{5n+1}}{2^{n+1}(5n+1)}\right)=\sum_{n=0}^{\infty}(-1)^{5n+1}\cdot\frac{2^{\frac{1}{5}}}{2}\cdot\frac{1}{5n+1}$ converges by AST.

$\therefore I=[-2^{\frac{1}{5}},2^{\frac{1}{5}})$

We can use differentation to find another series: e^x.

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$$

Proof

We know $R=\infty$ for that series. Let $g(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Then $g'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\underset{\text{*reindex}}{\sum_{n=0}^{\infty}}\frac{x^n}{n!}=g(x)$. So $g'(x)=g(x)$.

Solve this ODE and we get $g(x)=Ce^x$, but by definition $g(0)=1$, so $C=1$.

$\therefore g(x)=e^x$ ■

6.5 & 6.6 Review of Taylor Polynomials

Def: Taylor Polynomials

If f is n-times differentiable, then the n-th degree Taylor Polynomial for centered at $x=a$ is

$$T_{n,a}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2}+\dots\frac{f^{(n)}(a)(x-a)^n}{n!}$$

Def: Taylor Remainders

If f is n-times differentiable, then the n-th degree Taylor Remainder for centered at $x=a$ is

$$R_{n,a}(x)=f(x)-T_{n,a}(x)$$

The error in using $T_{n,a}(x)$ to approximate $f(x)$ is

$$\text{Error}=|R_{n,a}(x)|$$

Thm: Taylor's Theorem

Assume f is n+1 times differentiable on an interval I containing $x=a$.
Let $x\in I$. Then $\exists$ a point c between x and a $\ni$

$$f(x)-T_{n,a}(x)=R_{n,a}(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Corollary: Taylor's Inequality

$|R_{n,a}(x)|\le\frac{M|x-a|^{n+1}}{(n+1)!}$ where $|f^{(n+1)}(c)|\le M$ $\forall$ c between x and a.

6.7 & 6.8 Taylor Series and Convergence

Suppose $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+\dots$ for $|x-a|<R$ for $R>0$.
At $x=a$.
$f(a)=a_0$
$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+\dots\Rightarrow f'(a)=a_1$
$f''(x)=2a_2(x-a)+6a_3(x-a)+12a_4(x-a)^2+\dots\Rightarrow f''(a)=2a_2\Rightarrow a_2=\frac{f''(a)}{2}$
In general: $a_n=\frac{f^{(n)}(a)}{n!}$

Thm:

If $f(x)$ has a power series representation about $x=a$.
Say $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ for $|x-a|<R$, $R>0$, then

$$a_n=\frac{f^{(n)}(a)}{n!}$$

That is, $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ is the Taylor Series for f centered at $x=a$.

If $a=0$, then $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ is the Maclaurin Series for f.

Strength: The theorem says if find a series for a function,
- Manipulating known series.
- Differentiating/Integrating known series.
- Using the Taylor Series formula.

you will get the Taylor Series for f.

Weakness: The theorem assumes f has a power series and concludes it must be the Taylor Series. It doesn't say every function is equal to its Taylor Series.

Ex. Find the Maclaurin series for $f(x)=\begin{cases} 1/e & \text{if } x<-1 \\ e^x & \text{if } -1\le x\le 1 \\ e & \text{if } x>1 \end{cases}$

Since $f^{(n+1)}(0)=1\ \forall\, n\ge 0$, so the Maclauren Series is $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ which converges $\forall x\in\mathbb{R}$.
However $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \forall x\in\mathbb{R}$. But $f(3)=e\neq e^3=\sum_{n=0}^{\infty}\frac{3^n}{n!}$.
So $f(x)\neq\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \forall x\in\mathbb{R}$.

We need to determine if a function is equal to its Taylor Series on the interval of convergence.
Notice that the partial sums of a Taylor Series, $\sum_{n=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$, are the Taylor Polynomials, $\sum_{n=0}^{n}\frac{f^{(n)}(a)}{k!}(x-a)^k=T_{n,a}(x)$.
We want to determine for which $x\in\mathbb{R}$

$$f(x)=\lim_{n\to\infty}T_{n,a}(x).$$

and since we know $f(x)=T_{n,a}(x)+R_{n,a}(x)$, we verify

$$\lim_{n\to\infty}R_{n,a}(x)=0.$$

for each x where $R_{n,a}(x)\to 0$.

Corollary: Taylor's Inequality

If $|f^{(n+1)}(x)|\le M\ \forall\ |x-a|\le d$, then

$$|R_{n,a}(x)|\le\frac{M|x-a|^{n+1}}{(n+1)!}\quad\text{for } |x-a|\le d.$$

Thm. Convergence Theorem for Taylor Series

Assume f has derivatives of all orders on an interval I containing $x=a$.
Assume $\exists M\in\mathbb{R}\ni|f^{(k)}(x)|\le M\ \forall k\in\mathbb{N}$ and $x\in I$ (bounded on the interval). Then,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n\quad\forall x\in I$$

Proof:

If $x=a$, then $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)$

Assume that $x_0\in I$ and $x_0\neq a$. Since $|f^{(k)}(x)|\le M\ \forall k\in\mathbb{N}\ \forall x\in I$.

Taylor's Inequality gives $0\le|R_{n,a}(x_0)|\le\frac{M|x_0-a|^{n+1}}{(n+1)!}$.

Since $\lim_{n\to\infty}\frac{M|x_0-a|^{n+1}}{(n+1)!}=M\lim_{n\to\infty}\frac{|x_0-a|^{n+1}}{(n+1)!}=0$ (using $\lim_{n\to\infty}\frac{x^n}{n!}=0$).

We have $\lim_{n\to\infty}|R_{n,a}(x)|=0$ by Squeeze Thm. ■

Corollary:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad\forall x\in\mathbb{R}.$$

Proof:
Choose $B>0$. Then for $f(x)=e^x$, $|f^{(n+1)}(x)|=e^x\le e^B$ on $[-B,B]$.
By Convergence Thm, $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \forall x\in[-B,B]$.
Since B was arbitrary, so $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \forall x\in\mathbb{R}$. ■

Corollary:

Both $\sin x$ and $\cos x$ are equal to their Maclaurin Series $\forall x\in\mathbb{R}$.

Proof:
Both $\sin x$ and $\cos x$ are infinitely differentiable on $\mathbb{R}$
and their $\pm\sin x$, $\pm\cos x$ satisfy $|\pm\sin x|,|\pm\cos x|\le 1\ \forall x\in\mathbb{R}$.
By Convergence Thm, they're equal to their Maclaurin Series.

Ex. Find the Maclaurin Series for $f(x)=\sin(x)$.

$f(x)=\sin x \Rightarrow f(0)=0, \quad n=0$

$f'(x)=\cos x \Rightarrow f'(0)=1, \quad n=1$

$f''(x)=-\sin x \Rightarrow f''(0)=0, \quad n=2$

$f'''(x)=-\cos x \Rightarrow f'''(0)=-1, \quad n=3$

$f^{(4)}(x)=\sin x \Rightarrow f^{(4)}(0)=0, \quad n=4$

$$\sin x = \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}.$$

Ex. Find the Maclaurin Series for $f(x)=\cos(x)$.

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+1)(-1)^n x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

no constant term for power series of $\sin x$ (keep $n=0$)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}.$$

Ex.1 Find Taylor Series for e^x about $x=-3$

(need power series of $(x-(-3))=x+3$

$$e^x = e^{x+3-3} = e^{-3}e^{x+3} = \frac{1}{e^3}\sum_{n=0}^{\infty}\frac{(x+3)^n}{n!} = \sum_{n=0}^{\infty}\frac{(x+3)^n}{e^3 n!} \quad \forall x \in \mathbb{R} \quad (R=\infty)$$

Ex.2 Find Maclaurin Series for $f(x)=x^5\cos x$

$$x^5\cos x = x^5\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n+5} \quad \forall x \in \mathbb{R} \quad (R=\infty)$$

Ex.3 Find Taylor Series for $f(x)=\sin x$ about $x=\frac{\pi}{4}$

$$\sin x = \sin\left[\left(x-\tfrac{\pi}{4}\right)+\tfrac{\pi}{4}\right] = \sin\left(x-\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{4}\right) + \cos\left(x-\tfrac{\pi}{4}\right)\sin\left(\tfrac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} + \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{4}\right)^{2n}}{(2n)!}$$

$$= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor \frac{n}{2} \rfloor}\left(x-\frac{\pi}{4}\right)^n}{n!} \quad \forall x \in \mathbb{R} \quad (R=\infty)$$

6.9 Binomial Series

We know the Binomial Theorem for $(1+x)^k$ where $k \in \mathbb{N}$:

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n \text{ where } \binom{k}{n} = \frac{k!}{n!(k-n)!}$$

We can extend $k \in \mathbb{R}$ and find its Maclaurin Series.

$f(x) = (1+x)^k \Rightarrow f(0) = 1$

$f'(x) = k(1+x)^{k-1} \Rightarrow f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \Rightarrow f''(0) = k(k-1)$

$f^{(n)}(x) = k(k-1)\dots(k-(n-1))(1+x)^{x-n} \Rightarrow f^{(n)}(0) = k(k-1)\dots(k-n+1)$

We get $\sum_{n=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n$ for the Maclaurin Series.

The Radius of Convergence for $k \neq 0, 1, 2 \dots$:

$$\lim_{n\to\infty} \left| \frac{k(k-1)\dots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\dots(k-n+1)x^n} \right| = \lim_{n\to\infty} \left| \frac{k-n}{n+1} \right| |x| = |x|$$

We need $|x| < 1$, so $R = 1$ and the open interval is $(-1, 1)$.

The Interval of Convergence:

- If $k > 0$, $k \notin \mathbb{N}$, then $I = [-1, 1]$.
- If $-1 < k < 0$, then $I = (-1, 1]$.
- If $k \leq -1$, then $I = (-1, 1)$.
- If $k = 0, 1, 2, \dots$, then $I = \mathbb{R}$.

Notation: $\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!}$ ← n-terms in the numerator. keep in mind that $\binom{k}{0} = 1$. are called the Binomial Coefficients.

First, we claim $\binom{k}{n+1}(n+1) + \binom{k}{n}n = \binom{k}{n}k$ for $n \geq 1$.

$$\binom{k}{n+1}(n+1) + \binom{k}{n}n = \frac{k(k-1)\dots(k-n+1)(k-n)}{(n+1)!}(n+1) + \frac{k(k-1)\dots(k-n+1)}{n!}n$$

$$= \frac{k(k-1)\dots(k-n+1)}{n!}(k-n) + \frac{k(k-1)\dots(k-n+1)}{n!}n$$

$$= \binom{k}{n}(k-n) + \binom{k}{n}n$$

$$= \binom{k}{n}(k-n+n)$$

$$= \binom{k}{n}k$$

Let $f(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Next, we claim $f'(x) + xf'(x) = kf(x)$ $\forall x \in (-1, 1)$.

$$f'(x) + xf'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^n$$

$$= \binom{k}{1} + \sum_{n=2}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^n$$

$$= \binom{k}{1} + \sum_{n=1}^{\infty} \binom{k}{n+1}(n+1)x^n + \sum_{n=1}^{\infty} \binom{k}{n} n x^n \quad \leftarrow \text{re-index } 1^{st} \text{ sum}$$

$$= \binom{k}{1} + \sum_{n=1}^{\infty} \left(\binom{k}{n+1}(n+1) + \binom{k}{n}n \right) x^n$$

$$= \binom{k}{1} + \sum_{n=1}^{\infty} \binom{k}{n} k x^n \quad \leftarrow \text{by previous claim}$$

$$= k + \sum_{n=1}^{\infty} \binom{k}{n} k x^n$$

$$= k\left(1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n\right)$$

$$= k \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= kf(x)$$

Let $g(x) = \frac{f(x)}{(1+x)^k}$. Finally, show $g'(x) = 0$ for $x \in (-1, 1)$.

$$g'(x) = \frac{f'(x)(1+x)^k - f(x)k(1+x)^{k-1}}{(1+x)^{2k}}$$

$$= \frac{f'(x)(1+x)^k - (1+x)f'(x)(1+x)^{k-1}}{(1+x)^{2k}} \quad \leftarrow \text{by previous claim}$$

$$= \frac{f'(x)(1+x)^k - f'(x)(1+x)^k}{(1+x)^{2k}}$$

$$= 0$$

So $g'(x) = 0$ $\forall x \in (-1, 1)$, which means g is constant on $(-1, 1)$. Since $f(0) = 1$, we get $g(0) = \frac{1}{1} = 1$, so $g(x) = 1$ for $x \in (-1, 1)$. This implies $f(x) = (1+x)^k$ for $x \in (-1, 1)$

Thm: Generalized Binomial Theorem

Let $k \in \mathbb{R}$, then $\forall x \in (-1, 1)$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ where } \binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!}, \binom{k}{0} = 1.$$

Ex.1 Find the Maclaurin Series for arcsin(x).

First, find the Maclaurin Series for $\frac{1}{\sqrt{1+x}}=(1+x)^{-\frac{1}{2}}$.

$$(1+x)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-\frac{1}{2}-n+1)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n(1)(3)(5)\ldots(2n-1)}{2^n(n!)}x^n \text{ for } x\in(-1,1).$$

Next, find the Maclaurin Series for $\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$.

$$(1-x^2)^{-\frac{1}{2}}=(1+(-x^2))^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-\frac{1}{2}-n+1)}{n!}(-x^2)^n=\sum_{n=0}^{\infty}\frac{(1)(3)(5)\ldots(2n-1)}{2^n(n!)}x^{2n}$$

$|-x^2|<1 \Rightarrow |x|<1$ for $x\in(-1,1)$.

Finally, integrate.

$$\arcsin(x)=\sum_{n=0}^{\infty}\frac{(1)(3)(5)\ldots(2n-1)}{2^n(n!)(2n+1)}x^{2n+1} \text{ for } x\in(-1,1)$$

($C=0$ since $\arcsin(0)=0$).

6.10 Applications of Taylor Series

Recap of Known Series

- $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad (R=1)$
- $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (R=\infty)$
- $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (R=\infty)$
- $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \quad (R=\infty)$
- $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=\sum_{n=0}^{\infty}\frac{k(k-1)\ldots(k-n+1)}{n!}x^n \quad (R=1)$.

The applications we will examine are:

1. Finding Sums.
2. Evaluating Limits.
3. Evaluating & Approximating Integrals.

Finding Sums

Given a series, we may be able to manipulate it into one of the known series and find the sum that way. Alternatively, we could manipulate a known series into the given series.

Find the following sums:

Ex.1 $\sum_{n=0}^{\infty}\left(\frac{n+1}{n!}\right)x^n=S(x)$

Integrate: $\int S(x)dx=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}+C=x\sum_{n=0}^{\infty}\frac{x^n}{n!}+C=xe^x+C$

So $S(x)=\frac{d}{dx}\left(xe^x+C\right)=e^x+xe^x$

Ex.2 $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{2n+1}\right)+4=S(x)$

Differentiate: $S'(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n}=\sum_{n=0}^{\infty}(-x^2)^n=\frac{1}{1+x^2}$

So $S(x)=\int\frac{1}{1+x^2}dx=\arctan(x)+C$

But $S(0)=4$, so $C=4$. Thus $S(x)=\arctan(x)+4$.

Ex.3 $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{2^{2n}(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}=\cos\left(\frac{\pi}{2}\right)=0.$

Ex.4 Starting with $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, find $\sum_{n=0}^{\infty}\frac{nx^n}{7}$.

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\Rightarrow\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n-1}$$

$$\Rightarrow\frac{x}{(1-x)^2}=\sum_{n=0}^{\infty}nx^n$$

$$\Rightarrow\frac{x}{7(1-x)^2}=\sum_{n=0}^{\infty}\frac{nx^n}{7}$$

Ex.5 $\sum_{n=0}^{\infty}\frac{e(e-1)\ldots(e-n+1)}{3^n\cdot n!}=\sum_{n=0}^{\infty}\frac{e(e-1)\ldots(e-n+1)}{n!}\left(\frac{1}{3}\right)^n=\left(1+\frac{1}{3}\right)^e=\left(\frac{4}{3}\right)^e$

Evaluating Limits

We can use Taylor Series to evalute limits, instead of using l'Hopital's Rule. This idea is similar to how we used Taylor Polynomials and Taylor's Approximation Thm I to evalute limits.

Evaluate with series and not l'Hopital's Rule:

Ex. $\lim_{x\to 0} \frac{e^x-1}{x}$ $\quad e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$= \lim_{x\to 0} \frac{(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)-1}{x}$

$= \lim_{x\to 0} \frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+}{x}$

$= \lim_{x\to 0} 1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots = 1$

Ex.2 $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\quad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$

$= \lim_{x\to 0} \frac{1-(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots)}{x^2}$

$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\ldots}{x^2}$

$= \lim_{x\to 0} \frac{1}{2!}-\frac{x^2}{4!}+\ldots = \frac{1}{2}$

Ex.3 $\lim_{x\to 0} \frac{e^x-\frac{x^2}{2}-x-1}{\sin x - x}$ $\quad e^x = 1+x+\frac{x^2}{2!}+\ldots$ $\quad \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$

$= \lim_{x\to 0} \frac{(1+x+\frac{x^2}{2!}+\ldots)-\frac{x^2}{2}-x-1}{(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots)-x}$

$= \lim_{x\to 0} \frac{\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots}{-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots} \begin{matrix}\div x^3\\ \div x^3\end{matrix}$

$= \lim_{x\to 0} \frac{\frac{1}{3!}+\frac{x}{4!}+\ldots}{-\frac{1}{3!}+\frac{x^2}{5!}-\ldots} = \frac{\frac{1}{3!}}{-\frac{1}{3!}} = -1$

Evaluating Integrals as Series

Ex.1 Evaluate $\int e^{-x^2}dx$ as a series.

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} + C$$

Ex.2 How many terms would we need to use to approximate $\int_0^1 e^{-x^2}dx$ to an accuracy to $\frac{1}{10!(21)}$?

$$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

This converges by AST. Let's use AST estimation.

Write out some terms:

$$1-\frac{1}{1!(3)}+\frac{1}{2!(5)}-\frac{1}{3!(7)}+\frac{1}{4!(9)}-\frac{1}{5!(11)}+\frac{1}{6!(13)}-\frac{1}{7!(15)}+\frac{1}{8!(17)}-\frac{1}{9!(19)}+\frac{1}{10!(21)}$$ error

So, the estimate needs at least 10 terms.

Recap of Power Series

Strategy for Solving Questions

- Given a series, to find Radius and Interval of Convergence:
  - Ratio test for R and open interval.
  - Check endpoints with other tests.
- Given a series, to find its sum:
  - Relate it to a known series (e^x, $\sin x$, $\cos x$, $\frac{1}{1-x}$, $(1+x)^k$).
  - May need to integrate/differentiate.
- Given a function, to get its Taylor/Maclaurin Series, we can:
  - Use the Taylor Series formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ Find R and I.
  - Manipulate/Integrate/Differentiate a known series. R is know, but check endpoints to find I.
  - If asked for a Taylor Series about $x=a$, try $f(x) = f(x-a+a)$, manipulate and use a known series.