Q01. For this problem, you will need the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{and} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

In each case, find the integral using Riemann sums (that is, Theorem 1 on page 17) and not the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_0^4 x^2 + 2x \,\mathrm{d}x$

(b) $\displaystyle\int_1^3 2x^3 + x - 1 \,\mathrm{d}x$

Solution. We first note that both integrands are polynomial and therefore continuous. Therefore, we may apply the Integrability Theorem for Continuous Functions.

Using regular n-partitions, for (a), $t_i = a + i\Delta t = \frac{4i}{n}$. Then,

$$\begin{aligned}
\int_0^4 x^2 + 2x \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{4i}{n}\right)^2 + 2\left(\frac{4i}{n}\right)\right) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(\frac{16i^2}{n^2} + \frac{8i}{n}\right) \\
&= \lim_{n\to\infty} \frac{32}{n^2} \left(\frac{2}{n}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{32}{n^2} \left(\frac{(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty} \left(\frac{64n(n+1)(2n+1)}{6n^3} + \frac{16n(n+1)}{n^2}\right) \\
&= \frac{112}{3}
\end{aligned}$$

Likewise for (b), with regular n-partitions, $t_i = a + i\Delta t = 1 + \frac{2i}{n}$ and

$$\begin{aligned}
\int_1^3 2x^3 + x - 1 \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1 + \frac{2i}{n}\right)^3 + 1 + \frac{2i}{n} - 1\right) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(1 + \frac{6i}{n} + 3\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right)^3 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(1 + \frac{7i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \left(\sum_{i=1}^{n} 1 + \frac{7}{n}\sum_{i=1}^{n} i + \frac{12}{n^2}\sum_{i=1}^{n} i^2 + \frac{8}{n^3}\sum_{i=1}^{n} i^3\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \left(n + \frac{7(n+1)}{2} + \frac{12(n+1)(2n+1)}{6n} + \frac{8(n+1)^2}{4n}\right) \\
&= \lim_{n\to\infty} \left(4 + \frac{14(n+1)}{n} + \frac{8(n+1)(2n+1)}{n^2} + \frac{8(n+1)^2}{n^2}\right) \\
&= 42
\end{aligned}$$

where we simplify following the rules for infinite limits of rational functions. □

Q02. Express the following Riemann sums as definite integrals on the given interval, where x_i is a point in the i^{th} subinterval, and the partition is regular. Do not evaluate them.

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{x_i}\right)\frac{6}{n}$ on $[1,7]$.

(b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i+\ln(x_i+1)}\,\frac{3}{n}$ on $[2,5]$.

Solution. For (a), note that $\frac{b-a}{n}=\frac{6}{n}$. Therefore, $f(x_i)=\frac{e^{x_i}}{x_i}$.

Likewise for (b), we deduce that $f(x_i)=\sqrt{x_i+\ln(x_i+1)}$.

Notice that the target integrands, namely, $\frac{e^x}{x}$ and $\sqrt{x+\ln(x+1)}$ are continuous, we apply the Integrability Theorem for Continuous Functions in reverse. Therefore, we have both

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{x_i}\right)\frac{6}{n}=\int_1^7\frac{e^x}{x}\,\mathrm{d}x \qquad \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i+\ln(x_i+1)}\,\frac{3}{n}=\int_2^5\sqrt{x+\ln(x+1)}\,\mathrm{d}x \quad \square$$

Q03. Without evaluating the definite integral prove that

$$\frac{\sqrt{2}\pi}{24}\le\int_{\pi/6}^{\pi/4}\cos(x)\,\mathrm{d}x\le\frac{\sqrt{3}\pi}{24}$$

Proof. We apply Property 3 from Theorem 2. Notice that on the interval $[\frac{\pi}{4},\frac{\pi}{6}]$, we have that $\frac{\sqrt{2}}{2}\le\cos(x)\le\frac{\sqrt{3}}{2}$. Therefore, since $\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$, we have a lower bound of $\frac{\sqrt{2}}{2}(\frac{\pi}{12})=\frac{\sqrt{2}\pi}{24}$ and an upper bound of $\frac{\sqrt{3}}{2}(\frac{\pi}{12})=\frac{\sqrt{3}\pi}{24}$, as desired. $\square$

Q04. Consider the function

$$f(x)=\begin{cases}1 & \text{if } x \text{ is an irrational number}\\ \pi & \text{if } x \text{ is a rational number}\end{cases}$$

Prove that f is not integrable on $[0,1]$.

Proof. Recall that it is a property of the continuum that for any interval I, there exists both a rational number and an irrational number on I.

Let $P^{(n)}$ be the regular n-th partition of $[0,1]$.

Let S_n^1 be the Riemann sum associated with $P^{(n)}$ where c_i is irrational for all i. Then, $f(c_i)=1$ for all i, and S_n^1 traces out the rectangle bounded by the origin and $(1,1)$ with area 1.

Likewise, let S_n^2 be the Riemann sum associated with $P^{(n)}$ where c_i is rational for all i. Then, S_n^2 traces out the rectangle bounded by the origin and $(1,\pi)$ with area π.

Since these do not agree, by the definition of integrability, f is not integrable. $\square$

Q05. Let f be a continuous function. If $f_{ave}[a,b]$ denotes the average value of f on the interval $[a,b]$, and $a<c<b$, prove that

$$f_{ave}[a,b]=\frac{c-a}{b-a}f_{ave}[a,c]+\frac{b-c}{b-a}f_{ave}[c,b].$$

Proof. Recall the average value of f on $[a, b]$ is defined as $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$. Therefore, our desired statement is equivalent to

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &= \frac{c-a}{b-a}\left(\frac{1}{c-a}\int_a^c f(x)\,\mathrm{d}x\right) + \frac{b-c}{b-a}\left(\frac{1}{b-c}\int_c^b f(x)\,\mathrm{d}x\right) \\
\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &= \frac{1}{b-a}\left(\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\right) \\
\int_a^b f(x)\,\mathrm{d}x &= \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x
\end{aligned}$$

which is simply Theorem 3. □

Q06. If f is a continuous function and $\displaystyle\int_1^3 f(x)\,\mathrm{d}x = 8$, prove that there exists $c \in [1, 3]$ such that $f(c) = 4$.

Proof. First, notice the average value of f on $[1, 3]$ is $\displaystyle\frac{1}{3-1}\int_1^3 f(x)\,\mathrm{d}x = \frac{1}{2}(8) = 4.$

Then, since f is continuous, the conclusion follows from the Average Value Theorem. □