PMATH 370 Winter 2024:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Winter 2024 offering of PMATH 370, taught by Blake Madill.

Lectures

Chapter 1

Iteration and Orbits

1.1 Orbits

Definition 1.1.1 (iteration)
Let $f : A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$. For $a \in A$ we may iterate the function at a:

$$x_1 = a, x_2 = f(a), x_3 = \underbrace{f(f(a))}_{f^2(a)}, \dots, x_i = f^{i-1}(a), \dots .$$

The sequence $(x_n)_{n=1}^{\infty}$ is the orbit of a under f (abbreviated (x_n) without limits).

Lecture 1
Jan 8

Example 1.1.2. Let $f(x) = x^4 + 2x^2 - 2$, $a = -1$. What is the orbit of a under f?

Solution. $a = -1$, $f(a) = 1$, $f(f(a)) = f(1) = 1$, so we have $-1, 1, 1, 1, \dots$. We call this eventually constant. □

Example 1.1.3. Let $f(x) = -x^2 - x + 1$, $a = 0$. What is the orbit of a under f?

Solution. Calculate: $0, 1, -1, 1, -1, 1, \dots$. We call this eventually periodic (with period 2). □

Example 1.1.4. Let $f(x) = x^3 - 3x + 1$, $a = 1$. What is the orbit of a under f?

Solution. Calculate the first few terms: $1, -1, 3, 19, \dots$ (too big). This is a divergence to infinity. □

Example 1.1.5. Let $f(x) = x^2 + 2x$, $a = -0.5$. What is the orbit of a under f?

Solution. Calculate: $-0.5, -0.75, -0.9375, -0.9961 \dots$ and we make an educated guess that this converges to -1 since $f(-1) = -1$, a fixed point. □

Example 1.1.6. Let $f(x) = x^3 - 3x$, $a = 0.75$. What is the orbit of a under f?

Solution. Calculate: $0.75, -1.828, -0.625, 1.631, -0.552, \dots$. There is no clear pattern, so we call this chaotic. In fact, the orbit is dense in a neighbourhood of 0. □

We can start to formalize the examples.

Definition 1.1.7 (fixed point)

Let $f : A \to \mathbb{R}$ such that $f(A) \subseteq A$. A point $a \in A$ is fixed if $f(a) = a$.

Then, the orbit of a under f is $(a, a, a, \dots)$ which is constant.

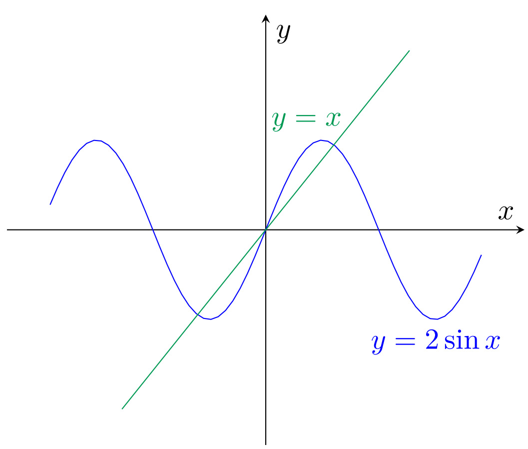
Example 1.1.8. Find all fixed points of $f(x) = x^2 + x - 4$.

Solution. We find points where $f(x) = x$, i.e., $x^2 + x - 4 = x$.

$$x^2 + x - 4 = x \iff x^2 = 4 \iff x = \pm 2$$ □

Example 1.1.9. How many fixed points does $f(x) = 2\sin x$ have?

Solution. Consider where the curve $y = 2\sin x$ meets $y = x$:



We can see there are three fixed points. □

Example 1.1.10. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Proof. We must show there is a solution to $x^4 - 3x + 1 \iff x^4 - 4x + 1 = 0$. Let $g(x) = x^4 - 4x + 1$. Since $g(x)$ is continuous, $g(0) = 1 > 0$, and $g(1) = -2 < 0$, by the Intermediate Value Theorem, there must exist a root of g on the interval $(0, 1)$. That is, a fixed point of f. □

Definition 1.1.11 (periodicity)

Let $f : A \to \mathbb{R}, f(A) \subseteq A$.

1. A point $a \in A$ is periodic for f if its orbit is periodic. An orbit is periodic if for some $n \in \mathbb{N}$, $f^n(a) = a$. The smallest n is the period of (the orbit of) a.
2. An orbit (of a point) is eventually periodic if there exists $n < m$ such that $f^n(a) = f^m(a)$. The smallest difference $m - n$ is the period of the orbit.

Lecture 2
Jan 10

Definition 1.1.12 (doubling function)

$D : [0, 1) \to [0, 1) : x \mapsto 2x - \lfloor 2x \rfloor$ returns the fractional part of $2x$.

Example 1.1.13. $D(0.4) = 0.8,\ D(0.6) = 0.2,\ D(0.8) = 0.6,\ D(0.5) = 0$.

This is a nice function that gives lots of periodic orbits for funsies.

Example 1.1.14. Find the orbit of $a = \frac{1}{5}$ under D.

Solution. Double until we pass 1: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5} \to \frac{3}{5}, \frac{6}{5} \to \frac{1}{5}$. The period is $\left|\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}\right| = 4$. □

Example 1.1.15. Find the orbit of $a = \frac{1}{20}$ under D.

Solution. Double: $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$ and we can stop because ex. 1.1.14 showed $\frac{1}{5}$ is periodic.

So this is eventually periodic with period 4. □

Problem 1.1.16

Given f and a, does $f^n(a)$ tend towards some limit L?

To solve this problem, we need to rigorously define "tend" and "limit".

1.2 Real analysis review

Notation. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, we write $(x_n) \subseteq \mathbb{R}$.

Definition 1.2.1 (convergence of a sequence)
Let $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$.

We say (x_n) converges to x if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Then, we write $x_n \to x$ or $\lim x_n = x$.

Example 1.2.2. Show that $\frac{1}{n} \to 0$.

Proof. Let $\varepsilon > 0$. Consider $N = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}$. For $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Therefore, $\frac{1}{n} \to 0$. □

Example 1.2.3. Prove that $\frac{2n}{n+3} \to 2$.

Proof. Let $\varepsilon > 0$. Since we know $\frac{1}{n} \to 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{6}$.

For $n \geq N$,

$$\left|\frac{2n}{n+3} - 2\right| = \left|\frac{2n}{n+3} - \frac{2n+6}{n+3}\right| = \left|\frac{-6}{n+3}\right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

Therefore, $\frac{2n}{n+3} \to 2$. □

Definition 1.2.4 (bounded sequence)
A sequence (x_n) is bounded (by M) if there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Proposition 1.2.5 (convergence implies boundedness)
If (x_n) is convergent, then (x_n) is bounded.

Proof. Suppose $x_n \to x$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x| < 1$.

For $n \geq N$, $|x_n| - |x| \leq |x_n - x| < 1$. That is, $|x_n| < 1 + |x|$.

Let $M = \max\{|x_1|, \dots, |x_{n-1}|, 1 + |x|\}$. Then, for both all $n < N$ and $n \geq N$, we have $|x_n| \leq M$. □

Remark 1.2.6. The converse is not true. Notice that $x_n = (-1)^n$ is bounded by 1 but obviously not convergent.

Proposition 1.2.7 (limit laws)

Let $x_n \to x$ and $y_n \to y$. Then:

(1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

Proof. (1) Let $\varepsilon > 0$. Then, since $x_n \to x$ and $y_n \to y$, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - x| < \frac{\varepsilon}{2}$ and $n \geq N_2 \implies |y_n - y| < \frac{\varepsilon}{2}$.

For $N = \max\{N_1, N_2\}$ and $n \geq N$,

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

That is, $x_n + y_n \to x + y$.

(2) Let $\varepsilon > 0$. Notice that:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \tag{$*$}$$

Since x_n is bounded, there exists $M > 0$ such that $|x_n| \leq M$ for all n.

Let $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 &\implies |x_n - x| \leq \frac{\varepsilon}{2(|y| + 1)} \text{ and} \\ n \geq N_2 &\implies |y_n - y| < \frac{\varepsilon}{2M}. \end{aligned}$$

Then, for $n \geq N := \max\{N_1, N_2\}$, $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $(*)$. ☐

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Jan 12

Definition 1.2.8 (Cauchy sequence)

We say $(x_n) \in \mathbb{R}$ is Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all n and m,

$$n, m \geq N \implies |x_n - x_m| < \varepsilon$$

Proposition 1.2.9

Every convergent sequence is Cauchy.

Proof. Intuitively: if the terms get arbitrarily close to some limit, they must get arbitrarily close to each other.

Formally: Let $x_n \to x$ be a convergent sequence and $\varepsilon > 0$. Since x_n converges, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$.

Then, when $n, m \geq N$, we have:

$$\begin{aligned}|x_n - x_m| &= |x_n - x_m + x - x| \\ &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

as desired. □

We take the following theorem from real analysis without proof.

Theorem 1.2.10 (completeness of $\mathbb{R}$)
A sequence is Cauchy if and only if it is convergent.

The big idea here: To prove (x_n) is Cauchy, you do not have to guess the limit first. That is, if you must prove convergence but do not care about the limit's value, prove that it is Cauchy instead.

Definition 1.2.11 (continuity of a function)
Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. We say f is continuous at a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

If f is continuous at all $a \in A$, we say it is continuous.

We also take this theorem from MATH 137 without proof.

Theorem 1.2.12
A function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if for all sequences $(x_n) \subseteq A$ with $x_n \to a$, we have $f(x_n) \to a$.

1.3 Orbits, revisited

Proposition 1.3.1
If $f : [a, b] \to [a, b]$ is continuous, then $f(x)$ has a fixed point.

Proof. We know by the domain and codomain that $f(a) \geq a$ and $f(b) \leq b$. This means $f(a) - a \geq 0$ and $f(b) - b \leq 0$. By the IVT on the continuous function $g(x) = f(x) - x$, we know there exists an $x \in [a, b]$ such that $g(x) = f(x) - x = 0 \iff f(x) = x$, i.e., x is a fixed point. □

Definition 1.3.2 (contraction)

Let $f : A \to \mathbb{R}, A \subseteq \mathbb{R}$. We say f is a contraction if there exists $C \in [0, 1)$ such that for all $x, a \in A$,

$$|f(x) - f(y)| \leq C|x - y|$$

This is just a Lipschitz function with Lipschitz constant less than 1.

Proposition 1.3.3

Contractions are continuous.

Proof. Let $\varepsilon > 0$. Suppose f is a contraction such that $|f(x) - f(y)| \leq C|x - y|$.

Consider $y \in A$. Let $\delta = \frac{\varepsilon}{C+1}$ and assume that $x \in A$ and $|x - y| < \delta$. But we have:

$$|f(x) - f(y)| \leq C|x - y| \leq C\delta < \varepsilon$$

as desired. □

Definition 1.3.4 (closure of an interval)

We say $A \in \mathbb{R}$ is closed if whenever $(x_n) \subseteq A$ with $x_n \to x$, then $x \in A$.

Example 1.3.5. $[a, b]$ is closed but $(0, 1]$ is not because $\frac{1}{n} \to 0 \notin (0, 1]$.

Theorem 1.3.6 (Banach contraction mapping theorem)

Suppose $A \subseteq \mathbb{R}$ is closed and $f : A \to A$ is a contraction. Then, there exists a unique fixed point $a \in A$ for f.

Moreover, for all $x \in A$, $f^n(x) \to a$.

Example 1.3.7. Analyze the orbit of $f : [0, 1] \to [0, 1]$, $f(x) = \frac{1}{3-x}$.

Solution. We can observe that $\frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$.

Also, $f'(x) = \frac{1}{(3-x)^2}$. Notice that $\frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$. So by the mean value theorem, for all $x, y \in [0, 1]$, there exists $c \in (0, 1)$ such that:

$$\begin{aligned} f(x) - f(y) &= f'(c)(x - y) \\ |f(x) - f(y)| &= |f'(c)| \cdot |x - y| \\ &\leq \frac{1}{4}|x - y| \end{aligned}$$

Then, identifying $C = \frac{1}{4}$, f is a contraction. Now,

$$\frac{1}{3-x} = x \iff 1 = 3x - x^2 \iff x^2 - 3x + 1 = 0 \iff x = \frac{3 \pm \sqrt{9-4}}{2} \iff x = \frac{3-\sqrt{5}}{2}$$

where we pick the negative root because we need $x \in [0,1]$.

Therefore, by the Banach contraction mapping theorem, for all $x \in [0,1]$, $f^n(x) \to \frac{3-\sqrt{5}}{2}$. □

Lecture 4
Jan 15

Definition 1.3.8
A sequence $(a_n) \subseteq \mathbb{R}$ is strongly-Cauchy if there exists $(\varepsilon_n) \subseteq [0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and for all n, $|a_n - a_{n+1}| < \varepsilon_n$.

Informally, "far enough along the sequence, the *neighbours* must get close". This is distinct from Cauchy, which is "far enough along the sequence, the *terms* must get close".

Remark 1.3.9 (assignment hint!)**.** Let $\sum_{n=1}^{\infty} a_n = L$. This means that $\sum_{k=1}^{n} a_k \xrightarrow{n\to\infty} L$.

That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\sum_{k=1}^{n} a_k - L\right| < \varepsilon$.

But $\left|\sum_{k=1}^{n} a_k - L\right| = \left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$.

We can now prove the Banach contraction mapping theorem.

Proof. Let $A \subseteq \mathbb{R}$ be closed and suppose there exists $f : A \to A$ and $C \in [0,1)$ such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y in A.

Fix $x_0 \in A$ and construct the orbit $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_n = f(x_{n-1}) = f^n(x_0)$.

For $n \in \mathbb{N}$, since f is a contraction,

$$\begin{aligned}
|x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\
&\leq C|x_n - x_{n-1}| \\
&= C|f(x_{n-1}) - f(x_{n-2})| \\
&\leq C^2|x_{n-1} - x_{n-2}| \\
&\vdots \\
&\leq C^n|x_1 - x_0|
\end{aligned}$$

Since $\sum_{n=1}^{\infty} C^n|x_1 - x_0| = |x_1 - x_0| \sum_{n=1}^{\infty} C^n$ is a convergent geometric series, we have that the sequence (x_n) is strongly-Cauchy.

Hence, by Assignment 1, $x_n \to a$ for some limit point $a \in A$ since A is closed.

Since f is continuous (prop. 1.3.3), we have that $f(x_n) \to f(a)$. By definition, $f(x_n) = x_{n+1}$, so $x_n \to f(a)$. But we already know $x_n \to a$, so $a = f(a)$. That is, a is a fixed point of f.

It remains to show uniqueness.

Suppose $a, b \in A$ such that $f(a) = a$ and $f(b) = b$.

$$\begin{aligned} |f(a) - f(b)| &\leq C|a - b| \\ |a - b| &\leq C|a - b| \end{aligned}$$

Since $C < 1$, we must have $|a - b| = 0$, that is, $a = b$. $\square$

Chapter 2

Graphical Analysis

2.1 Cobweb plots

Recall ex. 1.1.9. To visualize the orbit of a under f, we can:

1. Superimpose $y = f(x)$ over the line $y = x$.
2. Connect a vertical line $(a, a) - (a, f(a))$
3. Connect a horizontal line $(a, f(a)) - (f(a), f(a))$
4. Connect a vertical line $(f(a), f(a)) - (f(a), f(f(a)))$
5. Connect a horizontal line $(f(a), f(f(a))) - (f(f(a)), f(f(a)))$
   etc.

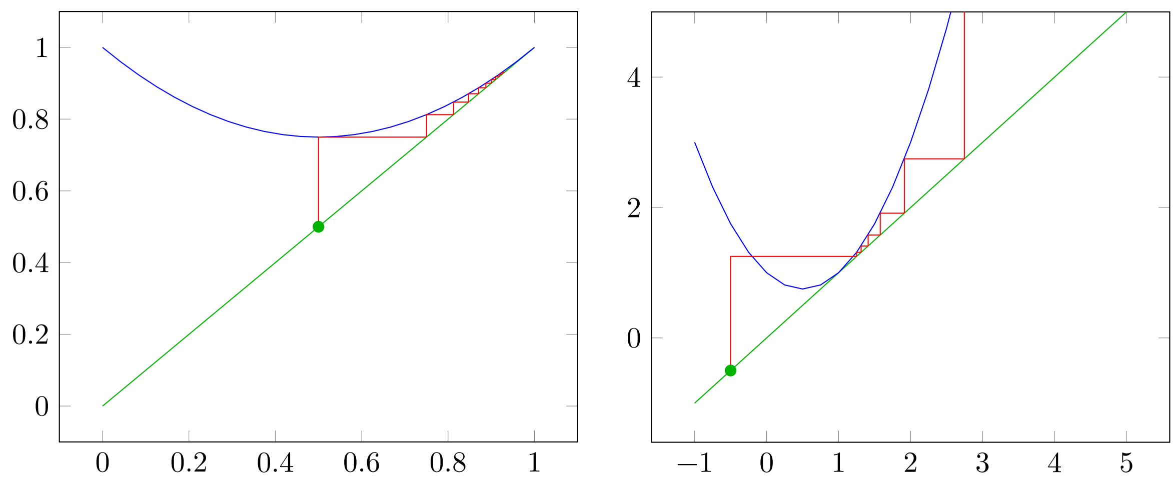
This is sometimes called a cobweb plot. We will be using https://marksmath.org/visualization/cobwebs/ to make cobweb plots.

Within these lecture notes, I use a LaTeX macro to draw plots defined here.

Example 2.1.1. Conduct a complete orbit analysis of $f(x) = x^2 - x + 1$

Solution. Playing around, we find that there is one fixed point $x = 1$.

When $x \in [0, 1]$, $f^n(x) \to 1$. Otherwise, $f^n(x) \to \infty$.



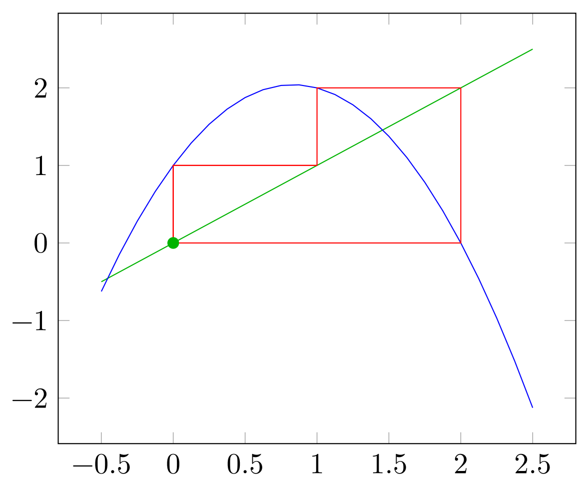
□

↓ *Lectures 5 and 6 adapted from Rosie* ↓

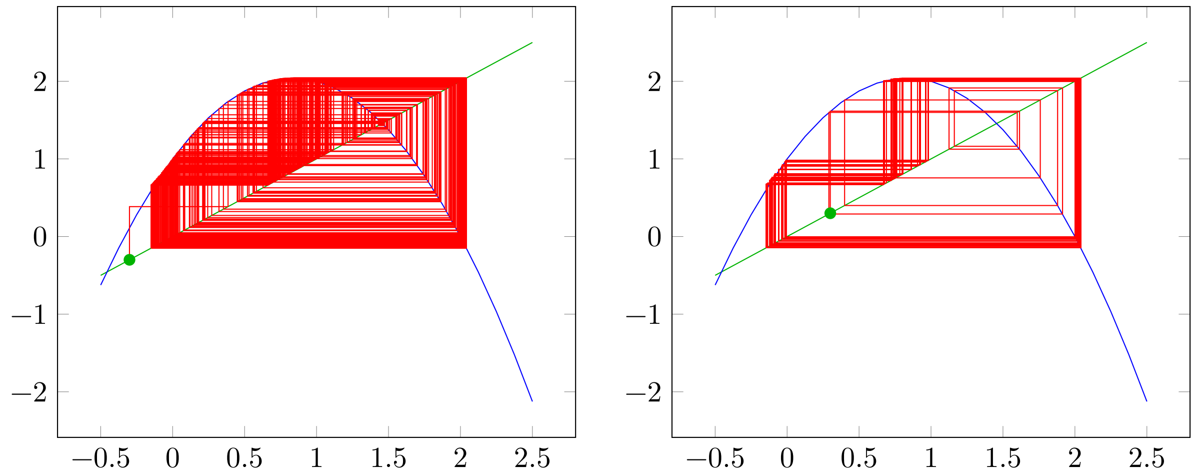
Lecture 5
Jan 17

Example 2.1.2. Conduct a complete orbit analysis of $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$.

Solution. At $x = 0$, we can see there is a cycle going from $0 \to 1 \to 2 \to 0$:



At points near 0, like $x = -0.3$ or $x = 0.3$, the graph becomes chaotic:



It appears that the cobweb densely covers the graph. □

As an aside, note that we cannot actually hit every point in the interval because the orbit is countable (i.e., has the same size as the naturals) but the interval is uncountable. We will later show that the points are dense (as the rationals are).

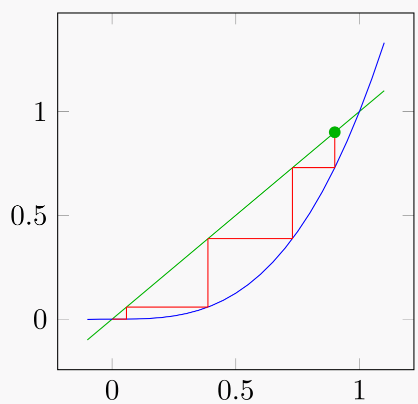
Chapter 3

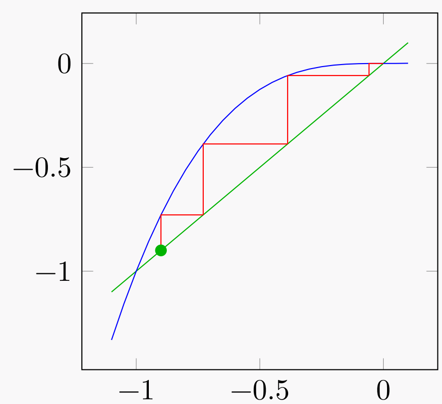
Fixed Points

3.1 Attracting/repelling fixed point theorems

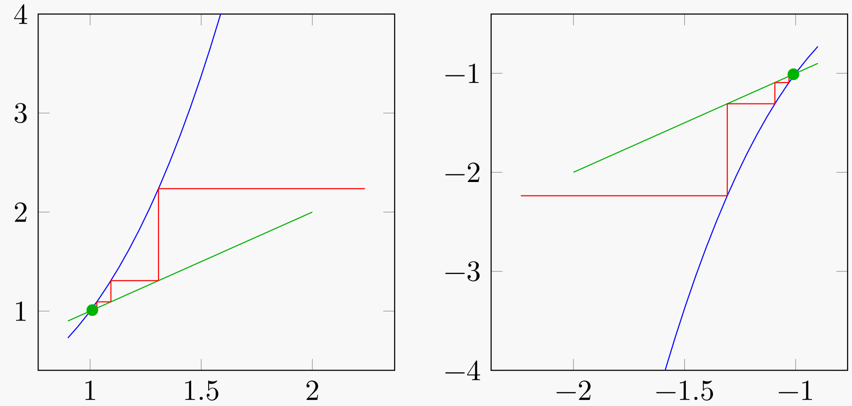
Remark 3.1.1. If $f(x)$ is continuous and $f^n(a) \to L$, then $f^{n+1}(a) \to f(L)$. Therefore, $f(L) = L$ is a fixed point.

Example 3.1.2. The function $f(x) = x^3$ has three fixed points: $0, \pm 1$. For $x \in (-1, 1)$, we see that $f^n(x) \to 0$:



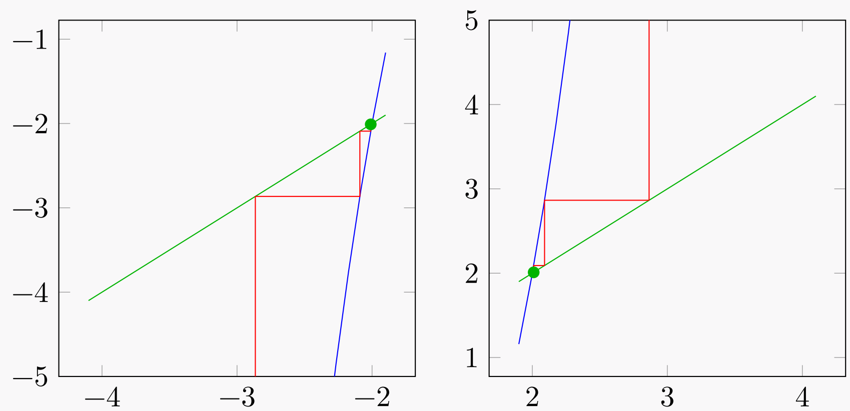


It looks like point 0 is attracting the orbit. For $x \in (-\infty, -1) \cup (1, \infty)$, we see $f^n(x) \to \infty$:

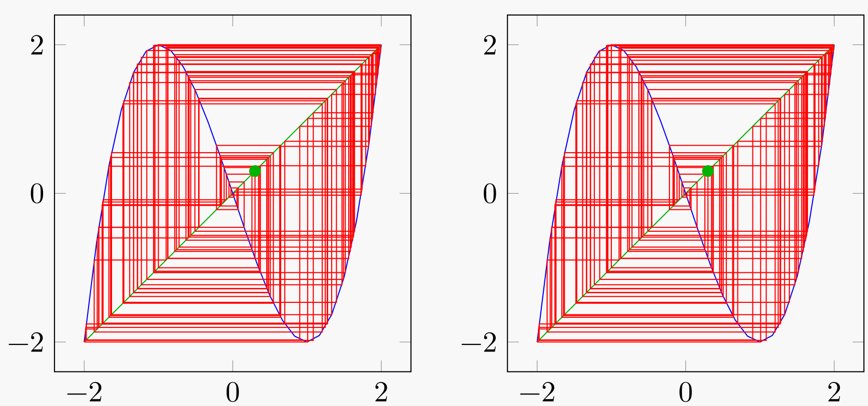


so the points ± 1 are <u>repelling</u> the orbit.

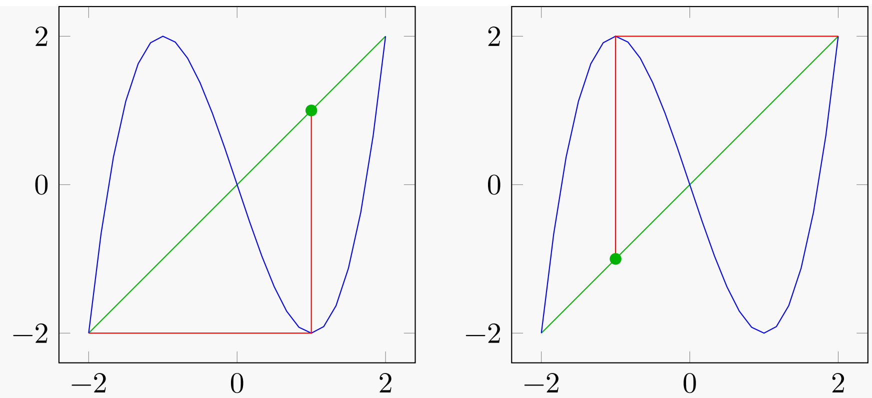
Example 3.1.3. The function $f(x) = x^3 - 3x$ also has three fixed points: $0, \pm 2$. To the right (left) of ± 2, orbits go to infinity:



The point 0 is repelling (in a different sense) since we get chaos:



At $x_0 = \pm 1$, the orbit is eventually constant, jumping to the fixed point ∓ 2:



Definition 3.1.4

Let a be a fixed point of $f(x)$.

1. If $|f'(a)| > 1$, we call a a repelling fixed point
2. If $|f'(a)| < 1$, we call a a attracting fixed point
3. If $|f'(a)| = 1$, we call a a neutral fixed point

Neutral fixed points can be a lot of different things.

Theorem 3.1.5 (attracting fixed point theorem)

Suppose a is an attracting fixed point of $f(x)$. Then, there exists an open interval I containing a such that

1. for all $x \in I$, $n \in \mathbb{N}$, $f^n(x) \in I$
2. for all $x \in I$, $f^n(x) \to a$

Recall the ε-δ definition of a limit.

Definition 3.1.6 (limit of a function at a point)

Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$.

We say a point $a \in A$ is non-isolated if for each $\varepsilon > 0$ there exists $b \in A$, $b \neq a$ with $b \in (a - \varepsilon, a + \varepsilon)$.

Suppose a is non-isolated. We say $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x - a| < \delta$.

It is important to define non-isolation. If a is isolated, we can choose a δ where $|x - a| < \delta$ is false. Then, every point is vacuously a limit point.

We now give the proof of the attracting fixed point theorem:

Proof. Assume $|f'(a)| < 1$. Then, there exists $c \in \mathbb{R}$ such that $|f'(a)| < c < 1$. By definition of the derivative, this means we can write

$$\lim_{x\to a} \frac{|f(x)-f(a)|}{x-a} < c$$

and by the definition of the limit, we know there exists $\delta > 0$ such that

$$\frac{|f(x)-f(a)|}{|x-a|} \leq c, \quad \forall x \in (a-\delta, a+\delta)$$

Hence, for $x \in I := (a-\delta, a+\delta)$, we have $|f(x)-f(a)| \leq c|x-a|$ and f is a contraction.

Lecture 6
Jan 19

In particular, for $x \in I$, we have

$$\begin{aligned} |f(x)-a| &= |f(x)-f(a)| && (a \text{ is a fixed point}) \\ &\leq c|x-a| \leq |x-a| && (c \in (0,1)) \\ &< \delta \end{aligned}$$

That is, $f(x) \in (a-\delta, a+\delta) = I$. Continuing for the rest of the orbit, for all $n \in \mathbb{N}$,

$$|f^n(x)-a| \leq c^n|x-a| \leq |x-a| < \delta$$

so we also have $f^n(x) \in I$.

Finally, notice that $0 \leq |f^n(x)-a| \leq c^n|x-a|$ and $c^n|x-a| \to 0$ since $c \in (0,1)$. By the squeeze theorem, $|f^n(x)-a| \to 0$. □

Theorem 3.1.7 (repelling fixed point theorem)
Suppose a is a repelling fixed point for $f(x)$. Then, there exists an open interval I containing a such that for all $x \in I$, $x \neq a$, there exists $n \in \mathbb{N}$ such that $f^n(x) \notin I$.

Proof. Say $|f'(a)| > c > 1$. Then, as above, there exists a δ such that

$$\lim_{x\to a} \frac{|f(x)-f(a)|}{x-a} > c \implies |f(x)-f(a)| \geq c|x-a|$$

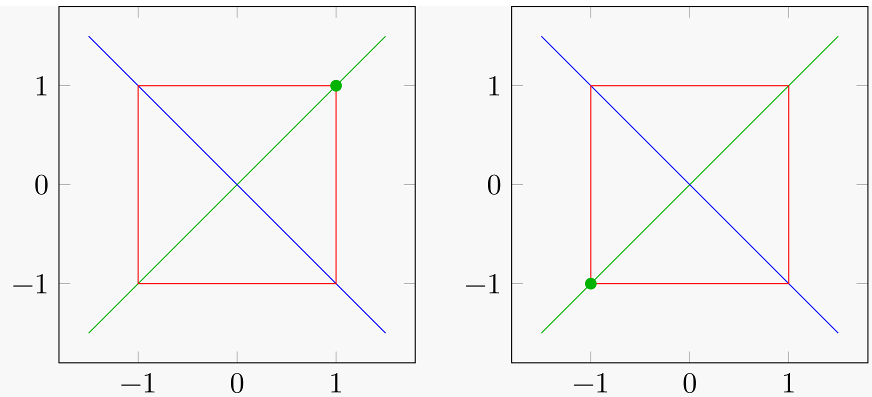
for all $x \in I := (a-\delta, a+\delta)$.

Since a is a fixed point, $|f(x)-f(a)| = |f(x)-a|$. Suppose for a contradiction that for all n, $f^n(x) \in I$. But since $c > 1$, $|f(n)-a| \geq c^n|x-a| \to \infty$. That is, δ must be arbitrarily large, which it is not. □

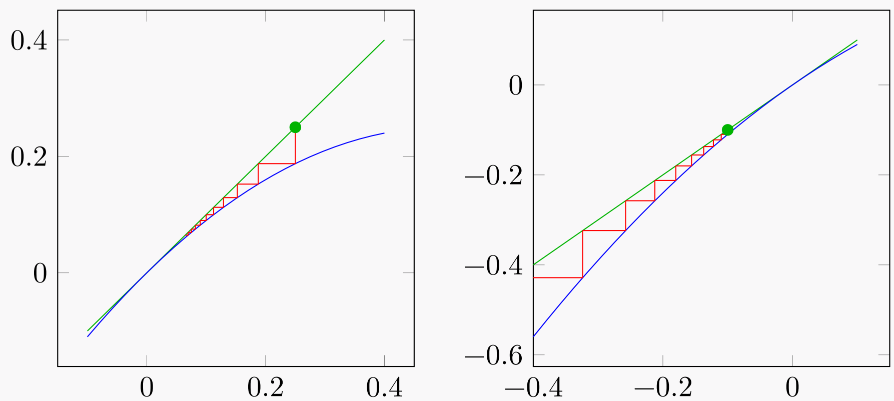
3.2 Neutral fixed points

Neutral fixed points can exhibit a lot of different behaviours.

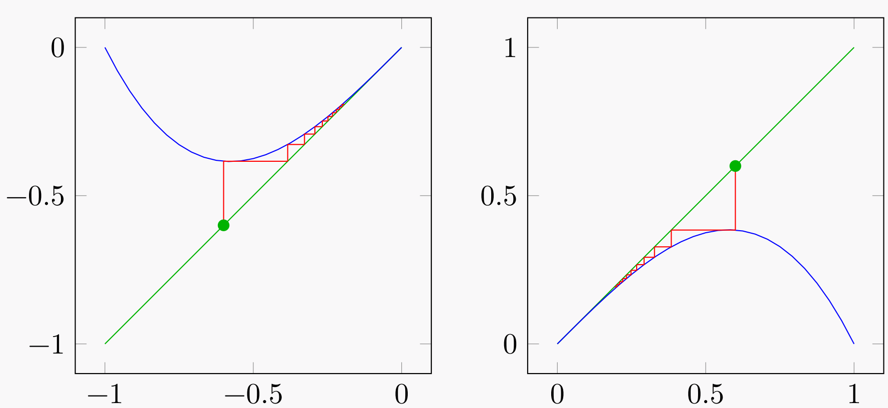
Example 3.2.1. For $f(x) = -x$, 0 is a fixed point with $|f'(0)| = 1$. The orbit bounces:



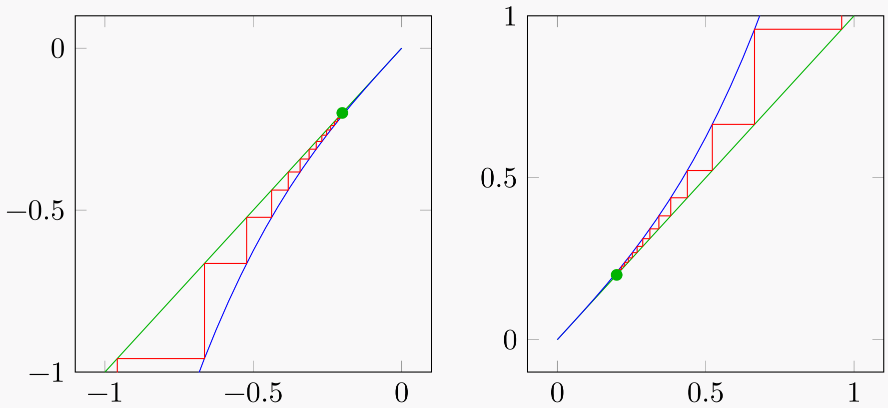
Example 3.2.2. For $f(x) = x - x^2$, $|f'(1)| = 1$ is a neutral fixed point. It is attracting from the right and repelling from the left:



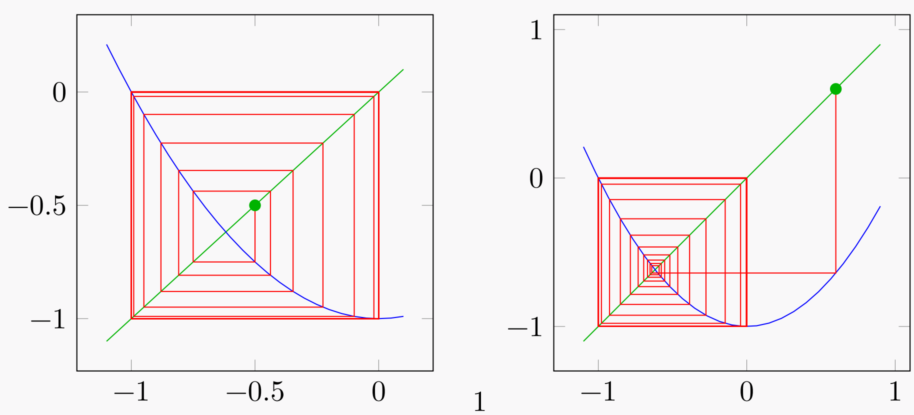
Example 3.2.3. For $f(x) = x - x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is <u>weakly attracting</u>, attracting but too slowly.



Example 3.2.4. For $f(x) = x + x^3$, $|f'(0)| = 1$ is a neutral fixed point. It is weakly repelling, repelling but too slowly:



Example 3.2.5. Consider $f(x) = x^2 - 1$. The orbit at $a = 0$ is periodic $(0, -1, 0, -1, \dots)$ with period 2. Near 0, the orbit tends to the $(0, -1)$-cycle:



We will call 0 an attracting periodic point because 0 is an attracting point of $f^2(x)$.

↑ *Lectures 5 and 6 adapted from Rosie* ↑

Lecture 7
Jan 22

Definition 3.2.6

Let a be a periodic point for $f(x)$ with period n.

We say a is an attracting/repelling/neutral periodic point if a is an attracting/repelling/neutral fixed point of f^n

Finding a closed form expression for something like $f^{10}(x)$ is a nightmare, so we need a better way.

Proposition 3.2.7

Let $f(x)$ be a differentiable function. Then, $(f^n)'(x) = f'(x) \cdot f'(f(x)) \cdots f'(f^{n-1}(x))$.

Proof. Proceed by induction on n.

If $n = 1$, we have $f'(x) = f'(x)$ and we are done.

Suppose $(f^n)'(x) = \prod_{k=0}^{n-1} f'(f^k(x))$ for some $n \geq 1$. Consider f^{n+1}:

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{n+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

by the chain rule. Then,

$$\begin{aligned} (f^{n+1})'(x) &= f'(f^n(x)) \cdot (f^n)'(x) \\ &= f'(f^n(x)) \cdot \prod_{k=0}^{n-1} f'(f^k(x)) \\ &= \prod_{k=0}^{n} f'(f^k(x)) \end{aligned}$$

completing the proof. □

Example 3.2.8. Analyze the periodic point $f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$, $a = 0$

Solution. The orbit is $(0, 1, 2, 0, 1, 2, \dots)$ with period 3.

We have $f'(x) = -3x + \frac{5}{2}$. Then, $(f^3)'(0) = f'(0)f'(1)f'(2) = (-\frac{7}{2})(-\frac{1}{2})(\frac{5}{2}) = \frac{35}{8} > 1$.

Therefore, the point is repelling. □

Chapter 4

Bifurcations

In general, bifurcation theory is the study of how a family of curves can change when a defining parameter is changed.

Consider the quadratic family:

$$Q_C(x) = x^2 + C$$

defined by the parameter $C \in \mathbb{R}$.

> **Problem 4.0.1**
>
> How does the behaviour (fixed points, orbits, etc.) of Q_C change based on C?

First, we can find the fixed points (if they exist) by solving

$$Q_C(x) = x \iff x^2 - x + C = 0 \iff x = \frac{1 \pm \sqrt{1-4C}}{2}$$

and note that $Q_C(x)$ has 2 fixed points when $C < \frac{1}{4}$, 1 fixed point when $C = \frac{1}{4}$, and no fixed points when $C > \frac{1}{4}$.

Suppose $C > \frac{1}{4}$. Then, we must have $Q_C^n(x) \to \infty$ for all x.

Instead, if $C = \frac{1}{4}$, $Q_C(x)$ has the unique fixed point $p = \frac{1}{2}$. Since $Q_C'(x) = 2x$ and $Q_C'(p) = 1$, this is a neutral fixed point. In fact, it attracts to one side and repels from the other.

Finally, if $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points $p_+ = \frac{1+\sqrt{1-4C}}{2}$ and $p_- = \frac{1-\sqrt{1-4C}}{2}$. Then, $Q_C'(p_+) =$

$1+\sqrt{1-4C}>1$ is repelling. Next,

$$\begin{aligned}
&-1 < Q_C'(p_-) < 1 \\
\iff &-1 < 1-\sqrt{1-4C} < 1 \\
\iff &-2 < -\sqrt{1-4C} < 0 \\
\iff &0 < \sqrt{1-4C} < 2 \\
\iff &-\frac{3}{4} < C < \frac{1}{4}
\end{aligned}$$

and in fact if $C < -\frac{3}{4}$, $Q_C'(p_-) < -1$ and if $C = -\frac{3}{4}$, $Q_C'(p_-) = -1$.

Theorem 4.0.2

For the family

$$Q_C(x) = x^2 + C,$$

depending on C:

1. All orbits tend to ∞ if $C > \frac{1}{4}$.
2. When $C = \frac{1}{4}$, $Q_C(x)$ has a unique fixed point $\frac{1}{2}$ and it is neutral.
3. If $C < \frac{1}{4}$, $Q_C(x)$ has two fixed points p_+ and p_-. The point p_+ is repelling. Moreover,
   (a) if $-\frac{3}{4} < C < \frac{1}{4}$, p_- is attracting;
   (b) if $C = -\frac{3}{4}$, p_- is neutral; and
   (c) if $C < -\frac{3}{4}$, p_- is repelling.

Lecture 8
Jan 24

Definition 4.0.3 (bifurcation)

We say a family of functions $F_\lambda(x)$ undergoes a bifurcation at λ_0 if there is a change in fixed point structure at λ_0.

Example 4.0.4. The quadratic family $Q_C(x) = x^2 + C$ undergoes a bifurcation at $\lambda_0 = \frac{1}{4}$.

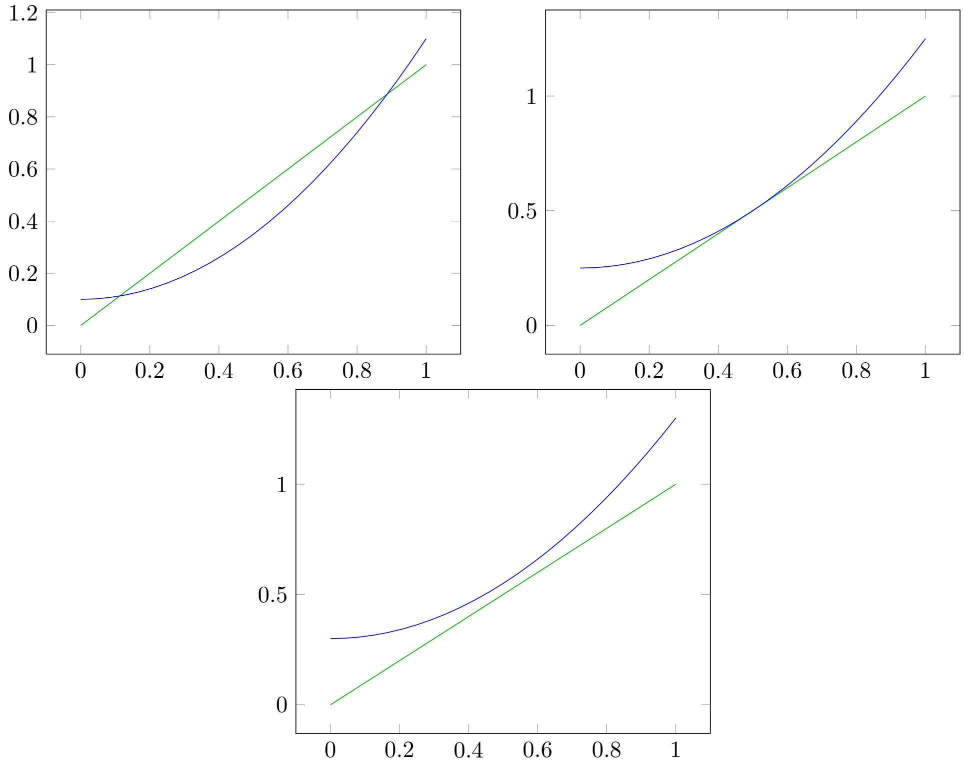
Definition 4.0.5 (tangent bifurcation)

A family $F_\lambda(x)$ undergoes a tangent bifurcation at λ_0 if there is an open interval I and an $\varepsilon > 0$ such that:

1. for $\lambda_0 - \varepsilon < \lambda < \lambda_0$, $F_\lambda(x)$ has no fixed points on I;
2. for $\lambda = \lambda_0$, $F_\lambda(x)$ has one fixed point and it is neutral; and
3. for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, $F_\lambda(x)$ has two fixed points in I, one of which is attracting and the other repelling.

(or with all inequalities flipped)

Visually, you have situations like



for $\lambda < \lambda_0$, $\lambda = \lambda_0$, and $\lambda > \lambda_0$.

Example 4.0.6. Consider the exponential family $E_\lambda(x) = e^x + \lambda$ at $\lambda_0 = -1$.

This is a tangent bifurcation.

Example 4.0.7. $F_\lambda(x) = \lambda x(1-x), \lambda_0 = 1$

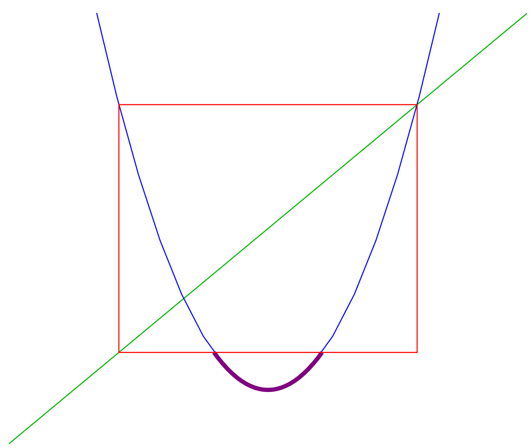
Here, we have two fixed points on one side of λ_0 and one fixed point on the other. So this is a bifurcation but not a tangent bifurcation.

Chapter 5

Cantor set

Recall the quadratic family $Q_C(x) = x^2 + C$ for $C < -2$. Then, $p_+ = \frac{1+\sqrt{1-4C}}{2} > 2$ and $-p_+ < -2$. Consider the interval/region $I = [-p_+, p_+]$ and $I \times I$.

Draw the picture of $y = x$, $y = Q_C(x)$, and the box $I \times I$:



Let $J_1 \subseteq I$ be the interval such that $Q_C(x) \notin I$ for all $x \in J_1$.

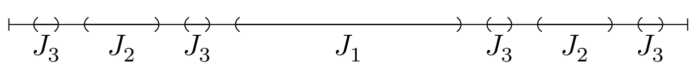
For $x \in J_1$, $Q_C^n(x) \to \infty$. Moreover, if there exists n such that $Q_C^n(x) \in J_1$, then $Q_C^n(x) \to \infty$.

Consider the set of points $\Lambda = \{x \in I : \forall n, Q_C^n(x) \in I\}$ with "interesting" orbits staying inside I.

Now, let $J_2 = \{x \in I : Q_C(x) \in J_1\} = \{x \in I : Q_C^2(x) \notin I\}$ and define higher J_n likewise.

Then, $\Lambda = I \setminus (J_1 \cup J_2 \cup \cdots)$ is a Cantor set, that is, a fractal. (roll credits!)

Drawing Λ on the x-axis, we get something that looks like



↓ *Lecture 9 adapted from Imaad* ↓

Lecture 9
Jan 26

Definition 5.0.1 (Cantor middle thirds set)
Let $C_0 = [0,1]$. Remove the open middle third interval each time.
That is, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on.
The set $K = \bigcap_{n=1}^{\infty} C_n$ is the Cantor (middle thirds) set.

Proposition 5.0.2
Suppose a bunch of sets $A_n \subseteq \mathbb{R}$ are closed. Then, $\bigcap A_n$ is also closed.

Proof. Let $(a_k) \subseteq \cap A_n$ where $(a_k) \to a$.
Note that for all n, $(a_k) \subseteq A_n \implies a \in A_n \implies a \in \bigcap A_n$ □

Proposition 5.0.3
Let $A, B \subseteq \mathbb{R}$ be closed. Then, $A \cup B$ is closed.

Proof. Let $(a_n) \subseteq A \cup B$ where $a_n \to a$.
WLOG, $\{n : a_n \in A\}$ is infinite. This allows us to construct $(b_n) \subseteq A$ such that $b_n \to a$.
Since A is closed, $a \in A \subseteq A \cup B$. □

Theorem 5.0.4 (Cantor sets are closed)
Any Cantor set, in particular K, is closed.

Theorem 5.0.5
K contains no non-empty open intervals.

Proof. Consider $I \subseteq K$. Then $\forall n, I \subseteq C_n$.
Then $\ell(I) \leq \frac{1}{3^n} \implies \ell(I) = 0 \implies I = \varnothing$, contradiction. □

Now, let's consider the base-3 expansion of $x \in [0,1]$. $x = 0.s_1 s_2 s_3, \cdots, s_i \in \{0,1,2\}$
Consider $\underbrace{[0,1/3]}_{s_1=0}$ and $\underbrace{[2/3,1]}_{s_1=2}$ and $\underbrace{[0,1/9]}_{s_1=0, s_2=0}$ $[2/9,1/3]$ $[2/3,7/9]$ $[8/9,1]$.

Remark 5.0.6. $x \in K$ if and only if x can be written in base 3 using only 0s and 2s

Example 5.0.7. $\frac{1}{3} \in K$. $\frac{1}{3} = 0.1_3 = 0.02222\ldots_3$

Theorem 5.0.8

K is uncountable and $|K| = |\mathbb{R}|$.

↑ *Lecture 9 adapted from Imaad* ↑

Chapter 6

Symbolic dynamics

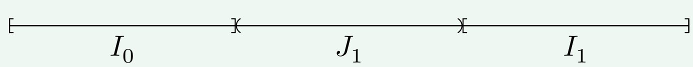
Lecture 10
Jan 29

Recall the construction of the Cantor set from the quadratic family:

> Fix $C < -2$ and consider $Q_C(x) = x^2 + C$. Define an interval $I = [-p_+, p_+]$ for a fixed point $p_+ = \frac{1+\sqrt{1-4C}}{2}$. Then, let
>
> $$\begin{aligned} J_1 &= \{x \in I : Q_C(x) \notin I\} \\ J_2 &= \{x \in I : Q_C(x) \in J_1\} \\ J_3 &= \{x \in I : Q_C(x) \in J_2\} \\ &\vdots \end{aligned}$$
>
> and define $\Lambda = I \setminus (\bigcup J_i) = \{x \in I : \forall n, Q_C^n(x) \in I\}$.

We proceed to do some analysis of Λ by translating into some sort of sequence space, doing analysis, and then going back to the Cantor set.

Notation. Define closed intervals $I_0 \cup I_1 := I \setminus J_1$ on the left/right of J_1:



Definition 6.0.1

For $x \in \Lambda$, the <u>itinerary</u> of x is the sequence $S(x) = (x_0x_1x_2x_3\cdots)$ with $x_i \in \{0,1\}$ where $x_i = 0 \iff Q_C^i(x) \in I_0$ and $x_i = 1 \iff Q_C^i(x) \in I_1$.

Our goal is to understand $S(x)$ better so that we can glean information about Λ.

Notation. Let $\Sigma = \{(x_0x_1x_2\cdots) : x_i \in \{0,1\}\}$ be the sequence space. Write elements of Σ as binary strings. Then, $S : \Lambda \to \Sigma$ is a function.

It would be helpful to define some PMATH 351/topology shit.

6.1 Intro to topology

Definition 6.1.1 (metric space)

Let X be a set. A function $d : X \times X \to [0, \infty)$ is a metric if

1. $d(x, y) = 0 \iff x = y$ (positive definiteness),
2. $d(x, y) = d(y, x)$ (symmetry), and
3. $d(x, y) \le d(x, z) + d(z, y)$ (triangle inequality).

The pair (X, d) is a metric space.

Once we have a metric space with a notion d of distance, we can adapt all our definitions from real analysis to an abstract space.

Example 6.1.2. The following are all metrics:

- $X = \mathbb{R}$, $d(x, y) = |x - y|$
- $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$
- For any set X, the discrete metric $d(x, y) = [x \neq y]$ (but is not particularly useful).
- For a subset $A \subseteq R$, $d(x, y) = |x - y|$ is a metric.

Extremely helpfully, we can define a metric on the sequence space.

Definition 6.1.3 (Cantor space)

Let $X = \Sigma$. Define $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} |x_i - y_i|$.

This is well-defined (converges) since $|x_i - y_i| \le 1$ and $\sum 2^{-i}$ converges.

Example 6.1.4. Let $x = (1111 \cdots)$ and $y = (1010 \cdots)$. Calculate $d(x, y)$.

Solution. By definition,

$$\begin{aligned} d(x, y) &= \sum_{i=0}^{\infty} \frac{x_i - y_i}{2^i} \\ &= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} && \text{(even indices cancel)} \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} \\ &= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}} \right) = \frac{1}{2} \left(\frac{4}{3} \right) = \frac{4}{6} = \frac{2}{3} \end{aligned}$$

□

We don't want to do this manual calculation every time.

Proposition 6.1.5

Let $x, y \in \Sigma$.

1. If $x_i = y_i$ for $i \leq n$, then $d(x,y) \leq \frac{1}{2^n}$.
2. If $d(x,y) < \frac{1}{2^n}$, then $x_i = y_i$ for $i \leq n$.

Proof. Suppose $x_i = y_i$ for $i \leq n$. Then, $d(x,y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k}$ since the first n terms will be 0 and $|x_i - y_i| \leq 1$. That is, $d(x,y) \leq \frac{1/2^{n+1}}{1-\frac{1}{2}} = \frac{1}{2^n}$.

Conversely, suppose $d(x,y) < \frac{1}{2^n}$ and for a contradiction that there exists $k \leq n$ where $x_k \neq y_k$. Then, there will be a $\frac{1}{2^k}$ term in the sum, so $d(x,y) \geq \frac{1}{2^k} \geq \frac{1}{2^n}$. Contradiction. □

Example 6.1.6. Let $x = (0000\cdots)$ and $y = (1000\cdots)$. Then, the distance is $\frac{1}{2^0} = 1$. However, $x_0 \neq y_0$.

Definition 6.1.7 (shift map)

The map $\sigma : \Sigma \to \Sigma : (x_0 x_1 x_2 \cdots) \mapsto (x_1 x_2 x_3 \cdots)$ that shifts a bitstring one bit to the left.

Remark 6.1.8. $\sigma^k(x_0 x_1 x_2 \cdots) = x_k x_{k+1} x_{k+2} \cdots$

Lecture 11
Jan 31

Definition 6.1.9 (continuity in metric spaces)

Suppose (X,d) and (Y,d') are (possibly distinct) metric spaces.

A function $f : X \to Y$ is <u>continuous at $y \in X$</u> if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$,

$$d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

We say f is <u>continuous</u> if it is continuous at every $y \in X$

Proposition 6.1.10

The shift map $\sigma : \Sigma \to \Sigma$ is continuous.

Proof. Fix $y = (y_0 y_1 y_2 \cdots) \in \Sigma$ and let $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$.

Consider $\delta = \frac{1}{2^{n+1}}$. Let $x = (x_0 x_1 x_2 \cdots) \in \Sigma$ such that $d(x,y) < \delta$.

Therefore, by prop. 6.1.5, $x_i = y_i$ for $i = 0, 1, \ldots, n+1$. Then, $\sigma(x) = (x_1 x_2 x_3 \cdots)$ and $\sigma(y) = (y_1 y_2 y_3 \cdots)$ agree for the first n terms.

Again by prop. 6.1.5, $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \varepsilon$. □

Definition 6.1.11 (convergence in metric spaces)

Let (X, d) be a metric space, $(x_n) \subseteq X$, and $x \in X$.

We say (x_n) converges to x ($x_n \to x$) if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \varepsilon.$$

Proposition 6.1.12 (sequential characterization of continuity in metric spaces)

Let (X, d) and (Y, d') be metric spaces and $f : X \to Y$. Then, f is continuous if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Definition 6.1.13 (homeomorphism)

Let (X, d) and (Y, d') be metric spaces. A function $f : X \to Y$ is a homeomorphism if

1. f is injective,
2. f is surjective,
3. f is continuous, and
4. f^{-1} is continuous.

Suppose $f : X \to Y$ is a homeomorphism. Then, if $(x_n) \subseteq X$ with $x_n \to x$, then $f(x_n) \to f(x)$.

Likewise, suppose $(y_n) \subseteq Y$ with $y_n \to y$. Say $y_n = f(x_n)$ and $y = f(x)$. Then, $f(x_n) \to f(x)$, so $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ and $x_n \to x$.

That is, f is a *relabelling* of X to Y. We think of X and Y as the "same metric space".

6.2 Revisiting the itinerary

Remark 6.2.1. Suppose we have $x \in \Lambda$ with $S(x) = (x_0 x_1 \cdots)$. Then, by definition, $x \in I_{x_0}$, $Q_c(x) \in I_{x_1}$, $Q_c^2(x) \in I_{x_2}$, etc. Therefore, $S(Q_c(x)) = (x_1 x_2 \cdots) = \sigma(S(x))$.

Iterating, $S(Q_c^n(x)) = \sigma^n(x)$.

Theorem 6.2.2

$S : \Lambda \to \Sigma$ is a homeomorphism.

We will prove this with some more tools. Recall from MATH 137:

Theorem 6.2.3 (monotone convergence theorem)

If $(a_n) \subseteq \mathbb{R}$ is increasing/decreasing and bounded, then (a_n) converges.

Instead of using this directly, we use a lemma:

Lemma 6.2.4 (nested intervals lemma)
If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ are closed intervals, then $\bigcap_{i=1}^{\infty} I_n \neq \varnothing$.

Proof. Let $I_k = [a_k, b_k]$.

That is, $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \cdots$.

Then, (a_n) is increasing and $(a_n) \subseteq [a_1, b_1]$. Likewise, (b_n) is decreasing and $(b_n) \subseteq [a_1, b_1]$. By the monotone convergence theorem, $a_n \to a$ and $b_n \to b$ for some limit points a and b.

Therefore (handwavey), $\varnothing \neq [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$. □

We will now prove thm. 6.2.2. It is true for $c < -2$, but we will show it for $c < -\frac{5+2\sqrt{5}}{4}$. *Lecture 12 Feb 2*

Proof. (injective) Suppose $x, y \in \Lambda$ with $S(x) = S(y)$ but $x \neq y$. Then, for all n, $Q_c^n(x)$ and $Q_c^n(y)$ live in the same I_0 or I_1. Recall from Assignment 2 that for all $x \in I \setminus J_1 = I_0 \cup I_1$, we have $|Q_c'(x)| \geq \mu > 1$. By the mean value theorem,

$$|Q_c(x) - Q_c(y)| \geq \mu |x - y|.$$

Since Q_c is injective on I_0 and I_1, we have that $Q_c(x) \neq Q_c(y)$. Thus,

$$\begin{gathered} \left|Q_c^2(x) - Q_c^2(y)\right| \geq \mu^2 |x - y| \\ \vdots \\ |Q_c^n(x) - Q_c^n(y)| \geq \mu^n |x - y| \end{gathered}$$

Since $\mu > 1$, we have $\mu^n |x - y| \to \infty$. However, $|Q_c^n(x) - Q_c^n(y)| \leq \max\{\ell(I_0), \ell(I_1)\}$, so it cannot blow up to infinity. Contradiction, so we have injectivity.

(surjective) Let $y = (y_0 y_1 \cdots) \in \Sigma$. For $n \in \mathbb{N}$, define

$$I_{y_0 y_1 \cdots y_n} := \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \ldots, Q_c^n(x) \in I_{y_n}\}.$$

It is enough to show there exists

$$x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \cdots y_n}$$

which would imply $S(x) = y$. Clearly, by definition, $I_{y_0} \supseteq I_{y_0 y_1} \supseteq I_{y_0 y_1 y_2} \supseteq \cdots$

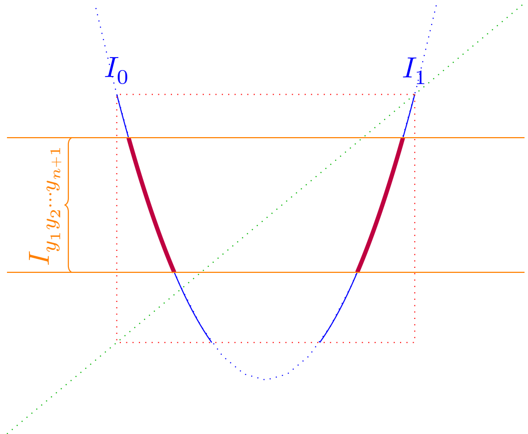
We claim that each $I_{y_0 y_1 \cdots y_n}$ is a closed interval. Proceed by induction.

First, $I_{y_0} \in \{I_0, I_1\}$ so it is closed. Assume $I_{y_0 y_1 \cdots y_n}$ is closed for some $n \geq 0$. Note:

$$\begin{aligned} & x \in I_{y_0 y_1 \cdots y_{n+1}} \\ \iff & x \in I_{y_0}, Q_c(x) \in I_{y_1}, Q_c(Q_c(x)) \in I_{y_2}, Q_c(Q_c^2(x)) \in I_{y_3}, \ldots, Q_c(Q_c^n(x)) \in I_{y_{n+1}} \\ \iff & x \in I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}}) \end{aligned} \tag{$\star$}$$

By the inductive hypothesis, $I_{y_1 y_2 \cdots y_{n+1}}$ is a closed interval (the subscript has length n).

We have



That is, $Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}})$ is a union of two disjoint closed intervals, one in I_0 and one in I_1.

In particular, returning to $(\star)$, $I_{y_0 y_1 \cdots y_{n+1}} = I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}})$ is one of these closed intervals.

By the nested intervals lemma, there must exist $x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \cdots y_n}$. Hence, $S(x) = y$ and we have surjectivity.

(continuous) Fix $y \in \Lambda$ and say $S(y) = (y_0 y_1 y_2 \cdots)$. Let $\varepsilon > 0$ and choose n such that $\frac{1}{2^n} < \varepsilon$.

Consider the 2^{n+1} disjoint, closed intervals $I_{t_0 t_1 \cdots t_n}$.

Pick $\delta > 0$ such that $(y - \delta, y + \delta)$ only overlaps with $I_{y_0 y_1 \cdots y_n}$. We know δ exists since we have a finite set of disjoint closed intervals.

For $x \in \Lambda$ with $|x - y| < \delta$, $x \in I_{y_0 y_1 \cdots y_n}$ and so $d(S(x), S(y)) \leq \frac{1}{2^n} < \varepsilon$.

(continuous inverse) Similar. □

Chapter 7

Chaos

7.1 Prerequisites to chaos

Lecture 13
Feb 5

Definition 7.1.1 (density)

Let (X, d) be a metric space. We say $A \subseteq X$ is dense in X if for all $x \in X$ and $\varepsilon > 0$, there exists $a \in A$ such that $d(a, x) < \varepsilon$.

Informally, there is always something "that close" to any point.

Example 7.1.2. $\mathbb{Q}$ is dense in $\mathbb{R}$. Given a real number, there is always a decimal approximation with arbitrary accuracy.

$\mathbb{Z}$ is not dense in $\mathbb{R}$. Given $x = \frac{1}{2} \in \mathbb{R}$, there are no integers within $\varepsilon = \frac{1}{4}$.

Example 7.1.3. Let $A = \{x \in \Sigma : \exists N, \forall i > N, x_i = 0\}$, i.e., the sequences which are eventually constant 0s. This is dense in Σ.

Proof. Let $x = (x_0 x_1 x_2 \cdots) \in \Sigma$ and let $\varepsilon > 0$. As usual, take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$.

Consider $y = (x_0 x_1 x_2 \cdots x_n 0000 \cdots) \in A$. Then, by prop. 6.1.5, $d(x, y) \leq \frac{1}{2^n} < \varepsilon$. □

Exercise 7.1.4. Let $A = \{x \in \Sigma : x \text{ is periodic}\}$. Show that this is dense in Σ.

Remark 7.1.5. A in exercise 7.1.4 is exactly the set of periodic points for the shift map $\sigma : \Sigma \to \Sigma$.

Proposition 7.1.6

There exists $z \in \Sigma$ such that $\{\sigma^k(z) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Σ.

Proof. Take $z = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ \cdots)$ to contain all possible blocks of increasing sizes.

Let $x \in \Sigma$ and $\varepsilon > 0$. Again, let $\frac{1}{2^n} < \varepsilon$.

For some k, $\sigma^k(z)$ and x agree on the first n terms. This must exist because z has *every possible* sequence of n terms. That is, by prop. 6.1.5, $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \varepsilon$. ☐

Warning: def. 7.1.7 is not the normal definition from applied math textbooks, but it is what we will use in the course.

Definition 7.1.7 (dynamical system)
A metric space (X, d) together with a continuous function $f : X \to X$.

This is an abstract space in which we can do orbit analysis and all our fun stuff.

> **Example 7.1.8.** $\sigma : \Sigma \to \Sigma$ is a dynamical system (see thm. 6.2.2).

Definition 7.1.9 (transitivity)
We say a dynamical system $f : X \to X$ is transitive if for all $x, y \in X$ and $\varepsilon > 0$, there exists $z \in X$ and $n, m \in \mathbb{N} \cup \{0\}$ such that $d(x, f^n(z)) < \varepsilon$ and $d(y, f^m(z)) < \varepsilon$.

Informally, given any two points, there is a special point whose orbit gets arbitrarily close to both points.

Proposition 7.1.10
$\sigma : \Sigma \to \Sigma$ is transitive.

Proof. Take z from prop. 7.1.6 such that the orbit is dense in Σ.

Then, for all $\varepsilon > 0$ and $x, y \in \Sigma$, there must exist by the definition of density n and m such that $d(x, \sigma^n(z)) < \varepsilon$ and $d(y, \sigma^m(z)) < \varepsilon$. ☐

Definition 7.1.11 (sensitive dependence on initial conditions)
Let $f : X \to X$ be a dynamical system.

We say f is sensitively dependent on initial conditions (or just sensitive) if

$$\exists \beta > 0,\ \forall \varepsilon > 0,\ \forall x \in X,\ \exists y \in X,\ \exists k \in \mathbb{N}$$

such that $d(x, y) < \varepsilon$ and $d(f^k(x), f^k(y)) \geq \beta$.

Informally, there exists a "wrongness" β that can always be achieved in the orbit no matter how close two starting points are.

Proposition 7.1.12

$\sigma : \Sigma \to \Sigma$ is sensitive.

Proof. Take $\beta = 1$.

Let $\varepsilon > 0$ and let $x \in \Sigma$. Say $\frac{1}{2^n} < \varepsilon$ and pick $y \in \Sigma$ such that $0 < d(x, y) < \frac{1}{2^n}$. That is, x and y must agree on the first n terms by prop. 6.1.5, but they are not equal.

Therefore, there exists $k \geq n$ such that $x_k \neq y_k$.

In the distance $d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^0} \geq 1 = \beta$. □

7.2 Defining chaos

↓ *Lectures 14 and 15 adapted from Imaad* ↓ *Lecture 14 Feb 7*

Definition 7.2.1 (chaos)

A dynamical system $f : X \to X$ is chaotic if

1. the periodic points for f are dense in X,
2. f is transitive, and
3. f is sensitive.

Theorem 7.2.2

$\sigma : \Sigma \to \Sigma$ is chaotic.

Proof. By props. 7.1.6, 7.1.10 and 7.1.12. □

Proposition 7.2.3

Let $(X, d), (Y, d')$ be metric spaces.

Suppose $f : X \to Y$ is continuous and surjective. If $A \subseteq X$ is dense in X, then $f(A)$ is dense in Y.

Proof. Let $y \in Y$ and say $y = f(x)$.

Let $\epsilon > 0$. Since f is continuous at x, there exists $\delta > 0$ such that

$$d(z, x) < \delta \implies d'(f(z), f(x)) < \epsilon$$

for any z. In particular, since A is dense in X, we may find $a \in A$ such that

$$d(a, x) < \delta \implies d'(f(a), f(x)) = d'(f(a), y) < \epsilon$$

as desired. □

Theorem 7.2.4
Let $c < \frac{-(5+2\sqrt{5})}{4}$. Then, $Q_c : \Lambda \to \Lambda$ is chaotic.

Proof. (periodic point density) Note that $Q_c^n(x) = x \iff S(Q_c^n(x)) = S(x) \iff \sigma^n(S(x)) = S(x)$.

By prop. 7.2.3 applied to $S^{-1} : \Sigma \to \Lambda$, the periodic points for Q_c are dense in Λ.

(transitivity) Take $z \in \Sigma$ from prop. 7.1.6 such that $\{\sigma^K(z) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Σ. Again by prop. 7.2.3, $\{S^{-1}(\sigma^K(z)) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Λ.

Note: Say $S(x) = z$, we know $(S(Q_c^K(x))) = \sigma^K(S(x)) \iff Q_c^K(x) = S^{-1}(\sigma^K(S(x)))$

This, $\{Q_c^K(x) : K \in \mathbb{N} \cup \{0\}\}$ is dense in Λ. As in prop. 7.1.10, we have that Q_c is transitive.

(sensitivity) Recall that $\Lambda \subseteq I \setminus J_1 = I_0 \cup I_1$. Let $\beta > 0$ be less than the gap between I_0 and I_1.

For $x, y \in \Lambda$ with $x \neq y$, supppose $S(x) \neq S(y)$. Then, there must exist a k where k^{th} term of $S(x)$ does not equal the k^{th} term of $S(y)$.

Hence, $|Q_c^k(x) - Q_c^k(y)| > \beta$ and Q_c is sensitive. □

Chapter 8

Sarkovskii's Theorem

Lecture 15
Feb 9

Theorem 8.0.1 (period 3)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a point with period 3, then f has a point with period n for all $n \in \mathbb{N}$.

Proposition 8.0.2

Let $I \subseteq J$ be closed intervals and suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. If $f(I) \supseteq J$, then $f(x)$ has a fixed point in I.

Proposition 8.0.3

Let I, J be closed intervals, $f : \mathbb{R} \to \mathbb{R}$ be continuous, and $f(I) \supseteq J$. Then, there exists a closed interval $I' \subseteq I$ such that $f(I') = J$.

We can now prove thm. 8.0.1.

Proof. Let $a \in \mathbb{R}$ be a period 3 point for $f(x)$. Say $f(a) = b, f(b) = c, f(c) = a$. WLOG, suppose $a < b$ and $a < c$.

Suppose $a < b < c$. The case where $a < c < b$ is left as an exercise.

Let $I = [a, b]$ and $J = [b, c]$. Then, $f(a) = b$ and $f(b) = c$ imply by IVT that $[b, c] = J \subseteq f(I)$. Likewise, $f(b) = c$ and $f(c) = a$ imply by IVT that $[a, c] = I \cup J \subseteq f(J)$.

Since $J \subseteq f(J)$, there exists a closed interval $A_1 \subseteq J$ such that $f(A_1) = J$ by prop. 8.0.3. Again, $A_1 \subseteq J = f(A_1)$, so there exists a closed interval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$.

Now, fix $n > 3$. Repeating the above process, we can find $A_{n-2} \subseteq A_{n-3} \subseteq \cdots \subseteq A_2 \subseteq A_1 \subseteq J$ such that $f(A_i) = A_{i-1}$. Now, $f(I) \supseteq J \supseteq A_{n-2}$ means there exists a closed interval $A_{n-1} \subseteq I$ such that $f(A_{n-1}) = A_{n-2}$.

Moreover, $f(J) \supseteq I \supseteq A_{n-1}$ which means there exists a closed interval $A_n \subseteq J$ such that $f(A_n) = A_{n-1}$.

We have $f^n(A_n) = J$ and $A_n \subseteq J$. By prop. 8.0.2, there exists $x_0 \in A_n$ such that $f^n(x_0) = x_0$.

Note: for $x_0 \in A_n$, $f(x_0) \in A_{n-1} \subseteq I$, $f^i(x_0) \in J$ for $i = 2, 3, \dots, n$.

For contradiction, suppose $f^i(x_0) = x_0$ for $i < n$.

Then, $\overbrace{f(x_0)}^{\in I} = \overbrace{f^{i+1}(x_0)}^{\in J} = b$ so $f(x_0) = b$, $f^2(x_0) = c$, and $f^3(x_0) = a$, which is a contradiction because $f^3(x_0) \in J$ but $a \notin J$. Hence, x_0 has period n.

That is, f has a periodic point with period n for all $n > 3$.

Further, $f(J) \supseteq J$ and so by prop. 8.0.2, f has a fixed point (aka period 1) in J.

Finally, $f(I) \supseteq J$ means $J = f(I')$ and $f(J) \supseteq I'$ means $f(J') = I'$. This implies $f^2(j') = f(I') = J \sup J'$. Therefore, we know there exists $x \in J'$ such that $f^2(x) = x$.

If $f(x) = x$, then $x \in J'$ and $f(x) \in I'$, meaning $x = b$. But, $f(b) \neq b = c$, contradiction.

Hence, x has period 2.

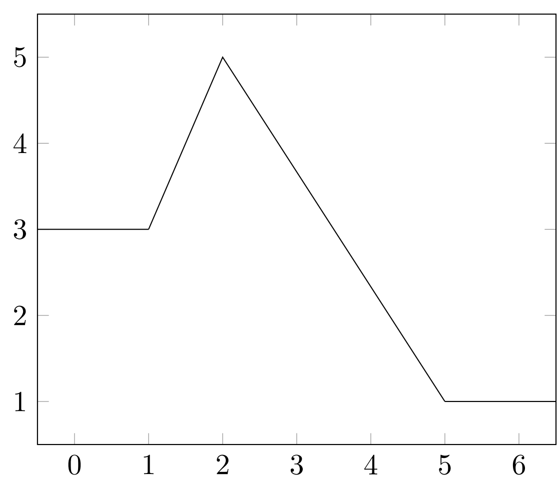
Therefore, since we already supposed f has a period 3 point, f has a period n point for all n. □

> **Exercise 8.0.4.** Complete the proof for the case where $a < c < b$.

↑ *Lectures 14 and 15 adapted from Imaad* ↑

Lecture 16
Feb 12

Draw the continuous function



Then, the orbit of 1 is $1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1$ and 1 has period 5.

> **Claim 8.0.5.** f has no point with period 3.

Proof. Suppose that f has a point x with period 3. Then, $1 \leq x \leq 5$.

Suppose $x \in [1, 2]$. Then, $x \in [1, 2] \cap f^3([1, 2])$ since $x = f^3(x)$. But $f^3([1, 2]) = [2, 5]$, so $x = 2$. However, 2 has period 5 since it is on the same 5-cycle given above.

Suppose instead that $x \in [2, 3]$. Then, $x \in [2, 3] \cap f^3([2, 3]) = [2, 3] \cap [3, 5] = \{3\}$ which is also on the 5-cycle.

If $x \in [4,5]$, then $x \in [4,5] \cap f^3([4,5]) = [4,5] \cap [1,4] = \{4\}$ which is, again, on the 5-cycle.

Finally, suppose that $x \in [3,4]$. Then, $f([3,4]) = [2,4]$ and it is strictly decreasing. Further, $f([2,4]) = [2,5]$ and it is also strictly decreasing. Once more, $f([2,5]) = [1,5]$ and it is again strictly decreasing. Since f^3 is strictly decreasing, it has a unique fixed point in $[3,4]$, but it is just the fixed point of f.

Since we have covered the entire interval $[1,5]$, x must not exist. ☐

Example 8.0.6. The function $f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$ has a period 1 point at $x = 0$, period 2 points $[-1,1] \setminus \{0\}$, and no other periodic points.

Definition 8.0.7 (Sarkovskii ordering)

Start by ordering the odd numbers $3 \prec 5 \prec 7 \prec 9 \prec \cdots$

Then, all those are $\cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \cdots$

All those are $\cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec \cdots$

Complete the ordering as $\cdots \prec 2^n \prec 2^{n-1} \prec \cdots \prec 2^2 \prec 2 \prec 1$.

This is a total order on the natural numbers.

Example 8.0.8.

- $26 = 2 \cdot 13 \prec 2^2 \cdot 5 = 40$ because the exponent of 2 is smaller.
- $3072 = 2^{10} \cdot 3 \prec 2^5 = 32$ because powers of 2 are big.
- $n \preccurlyeq 1$ for all n.
- $2^{15} \prec 2^3$ since the powers of 2 are ordered backwards.

Theorem 8.0.9 (Sarkovskii's theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $n \prec m$ in the Sarkovskii ordering. Then, if f has a point with period n, then it has a point with period m.

Chapter 9

Fractals

9.1 Definitions and dimensions

Lecture 17
Feb 14

Definition 9.1.1

Define a few things from topology.

- For $\mathbf{x} \in \mathbb{R}^n$, the norm $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2 + \cdots + x_n^2}$
- $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is our default metric on $\mathbb{R}^n$
- For $\mathbf{x} \in \mathbb{R}^n$, $\varepsilon > 0$, the open ball of radius ε centered at x is $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y} < \varepsilon\|\}$
- We say $U \in \mathbb{R}^n$ is open if for all $\mathbf{x} \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subseteq U$.
- The boundary $\delta(A)$ of a set $A \subseteq \mathbb{R}^n$ is the closure of A without the interior of A.

Definition 9.1.2 (topological dimension (zero case))

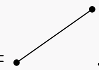
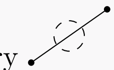
We say $S \subseteq \mathbb{R}^n$ has topological dimension $\dim_t S = 0$ if for all $\mathbf{x} \in S$, there exists arbitrarily small open sets $U \ni \mathbf{x}$ such that $\delta(U) \cap S = \varnothing$.

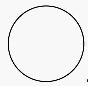
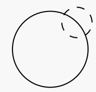
Example 9.1.3. Let $X =$ . Then, since we can draw balls  separating each point, $\dim_t X = 0$.

Example 9.1.4. $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ has topological dimension 0.

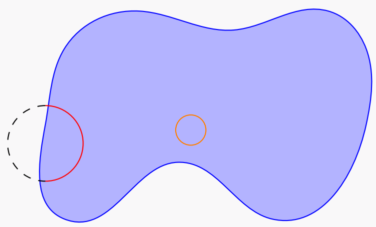
Definition 9.1.5 (topological dimension (non-zero case))

A set $S \subseteq \mathbb{R}^n$ has topological dimension $k \in \mathbb{N}$ if for all $\mathbf{x} \in S$, there exists arbitrarily small $U \ni \mathbf{x}$ such that $\delta(U) \cap S$ has topological dimension $k-1$, where k is minimal with this property.

Example 9.1.6. Consider a line $X =$ . Then, since any ball's boundary  creates an intersection made of two distinct points (i.e., a set with topological dimension 0), we know that $\dim_t X = 1$.

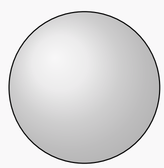
Example 9.1.7. Let X be a circle . Again, any ball's boundary  still only has two intersecting points, so $\dim_t X = 1$.

Example 9.1.8. Let X be a filled 2D region.



Then, the intersection of a ball's boundary will give either a circle or an arc, which have topological dimension 1, so the region has topological dimension 2.

Example 9.1.9. Let X be a non-filled sphere.



Then, the intersection of a 3D ball's boundary will give a circle, which has topological dimension 1, so $\dim_t X = 2$.

Example 9.1.10. Let X be a filled sphere.

Then, a 3D ball's boundary's intersection is either a hollow sphere or a spherical cap, which each have topological dimension 2, so $\dim_t X = 3$.

Definition 9.1.11 (fractal dimension)
We say $S \subseteq \mathbb{R}^n$ is self-similar if S may be divided into K congruent subsets, each of which may be magnified by a fixed M to yield S itself.

The fractal dimension of S is given by $\dim_f S = \frac{\ln K}{\ln M}$.

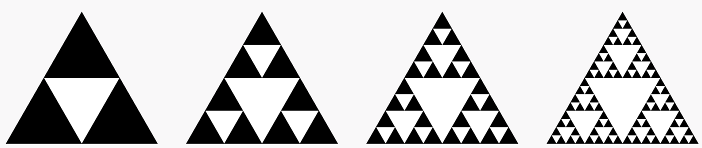
Definition 9.1.12 (fractal)
A fractal is a self-similar $S \subseteq \mathbb{R}^n$ such that $\dim_f S > \dim_t S$.

9.2 Fractal gallery

Example 9.2.1. Let $X =$ •——• be a line. Then, since we can divide it into n smaller lines each of size $\frac{1}{n}$, it has fractal dimension $\dim_f X = \frac{\ln n}{\ln n} = 1$. The topological dimension is $\dim_t X = 1$.

So this is not a fractal, and is indeed just boring (not a fractal).

Example 9.2.2 (Sierpinski triangle)**.** Let X be the Sierpinski triangle, i.e., the limiting point of the process:



Then, the topological dimension is $\dim_t X = 1$ because, in the limit, any ball will touch only single points. In particular, we can imagine balls touching the three points of a triangle.

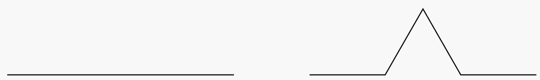
However, the fractal dimension is $\dim_f X = \frac{\ln 3}{\ln 2} \approx 1.58 > 1$ because each step is consisted of 3 previous steps scaled by $\frac{1}{2}$. so X is a fractal!

Lecture 18 Feb 16

Example 9.2.3 (Cantor set)**.** Let K be a middle-thirds Cantor set, i.e., the limiting point of the process:

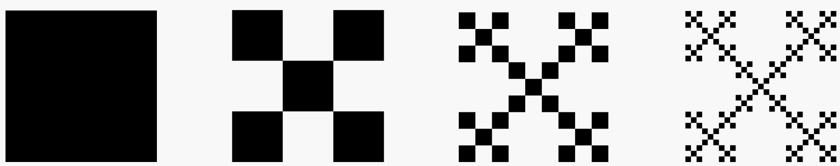
For any point in the Cantor set, we can find a small empty region around it since we keep cutting away from the sides. That is, $\dim_t K = 0$. However, $\dim_f K = \frac{\ln 2}{\ln 3} > 0$.

Example 9.2.4 (Koch curve)**.** Let X be the Koch curve, where each line segment is replaced by a bump:



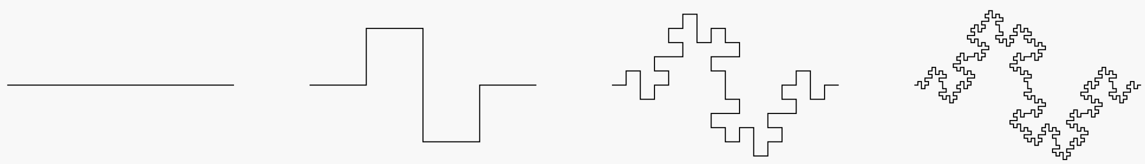
As a continuous line, intersection with a ball boundary gives points, so $\dim_t X = 1$. We have four copies scaled by $\frac{1}{3}$, so $\dim_t X = \frac{\ln 4}{\ln 3} > 1$.

Example 9.2.5 (box fractal)**.** Let X be a box fractal, where we delete edge pieces of a 3x3 grid:



Then, since the squares are solid, we have topological dimension 1 but fractal dimension $\dim_f X = \frac{\ln 5}{\ln 3} > 1$.

Example 9.2.6 (Minkowski sausage)**.** Let X be the Minkowski sausage, where each line segment is replaced by a square wave:



Then, as a continuous line, $\dim_t X = 1$, but we have $\dim_f X = \frac{\ln 8}{\ln 4} = \frac{3}{2} > 1$.

There is a hidden connection between iterated systems and fractals! For example, playing around with the website http://www.shodor.org/interactivate/activities/TheChaosGame/ has a process where each iteration moves a point halfway to one of the vertices.

...one reading week later... *Lecture 19 Feb 26*

Recall the chaos game:

1. Start with the vertices (v_1, v_2, v_3) of an equilateral triangle.
2. Pick $p \in_R \mathbb{R}^2$.
3. Pick $v_i \in_R \{v_1, v_2, v_3\}$.
4. Replace p with the midpoint of p and v_i.
5. Iterate.

Where does the orbit of p end up? Somehow, exactly in the Sierpinski triangle. Our goal is to formalize this.

9.3 Iterated function systems

Fix some $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and contraction factor $0 < \beta < 1$. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

i.e., $F(\mathbf{p}) = \beta(\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0$. Then,

1. $F(\mathbf{p}_0) = \mathbf{p}_0$
2. $\|F(\mathbf{p}) - F(\mathbf{p}_0)\| = \|\beta(\mathbf{p} - \mathbf{p}_0)\| = \beta\|\mathbf{p} - \mathbf{p}_0\|$
3. $\|F^n(\mathbf{p}) - \mathbf{p}_0\| = \beta^n\|\mathbf{p} - \mathbf{p}_0\| \to \mathbf{0}$ so $F^n(\mathbf{p}) \to \mathbf{p}_0$

Definition 9.3.1

Let $0 < \beta < 1$ and $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^2$. For each $i = 1, \ldots, n$, let

$$F_i(\mathbf{p}) = \beta(\mathbf{p} - \mathbf{p}_i) + \mathbf{p}_i$$

Then, $\{F_1, \ldots, F_n\}$ is an iterated function system (IFS).

Fix $\mathbf{q}_0 \in \mathbb{R}^2$. Randomly select an F_i. Let $\mathbf{q}_1 = F_i(\mathbf{q}_0)$. Repeat. The set of points in which the orbit $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots$ lives is the attractor for the IFS.

Example 9.3.2. Formalize the chaos game. Let $\mathbf{p}_1 = v_1$, $\mathbf{p}_2 = v_2$, $\mathbf{p}_3 = v_3$, and $\beta = \frac{1}{2}$. Then, $F_i(\mathbf{p}) = \frac{1}{2}(\mathbf{p} - \mathbf{p}_i) + \mathbf{p}_i = \frac{1}{2}(\mathbf{p} + \mathbf{p}_i)$ is the midpoint.

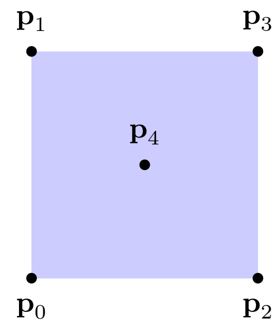
The set $\{F_1, F_2, F_3\}$ is an iterated function system whose attractor is the Sierpinski triangle.

Note that we can construct pathologically unlucky sequences of F_i's that give us point sequences that never reach the attractor. However, we ignore those :)

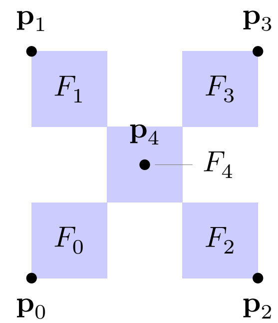
Example 9.3.3. Let $\mathbf{p}_0 = (0, 0)^\top$, $\mathbf{p}_1 = (1, 0)^\top$, $\mathbf{p}_2 = (0, 1)^\top$, $\mathbf{p}_3 = (1, 1)^\top$, $\mathbf{p}_4 = (\frac{1}{2}, \frac{1}{2})^\top$, and $\beta = \frac{1}{3}$.

What fractal does this produce?

Solution. Draw the points:



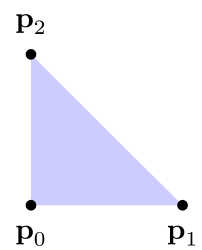
Divide the square into thirds (since we are using $\beta = \frac{1}{3}$). Then, colour in the images of the square under each F_i:



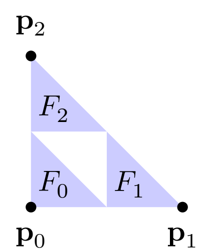
This is going to produce the box fractal. ☐

> **Example 9.3.4.** Repeat with $\mathbf{p}_0 = (0,0)^\top$, $\mathbf{p}_1 = (1,0)^\top$, $\mathbf{p}_2 = (0,1)^\top$, and $\beta = \frac{1}{2}$.

Solution. Again, draw the points:



Shrink the right triangle by a factor of $\beta = \frac{1}{2}$ around each point:



This will generate a Sierpinski-like triangle. □

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Feb 28

> **Example 9.3.5.** Let $\mathbf{p}_0 = (0,0)^\top$, $\mathbf{p}_1 = (1,0)^\top$, $\beta = \frac{1}{3}$. Repeat.

Solution. Write the functions explicitly

$$F_0(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x \\ \frac{1}{3}y \end{bmatrix} \quad \text{and} \quad F_1(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{bmatrix}$$

and pick a point $\mathbf{q}_0 = (x_0, y_0)^\top \in \mathbb{R}^2$. We say that the orbit of q_0 under $\{F_0, F_1\}$ is $q_0, q_1, q_2, \ldots$ with random selections $s_1, s_2, s_3, \ldots \in \{0,1\}$ where $q_i = F_{s_i}(q_{i-1})$.

First, notice that no matter which one we choose, $y_i = \frac{1}{3}y_{i-1}$. Therefore, $y_n = \frac{1}{3^n}y_0 \to 0$.

For the x-coordinate, we can write it out explicitly to find the pattern:

$$\begin{aligned} x_1 &= \frac{1}{3}x_0 + \frac{2s_1}{3} \\ x_2 &= \frac{1}{3^2}x_0 + \frac{2s_1}{3^2} + \frac{2s_2}{3} \\ &\vdots \\ x_n &= \frac{1}{3^n}x_0 + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \cdots + \frac{2s_n}{3} \end{aligned}$$

As $n \to \infty$, the first term disappears. The remaining term looks like a funny ternary expansion. Therefore, x_n gets arbitrarily close to points of the form $\sum_{i=1}^{\infty} \frac{t_i}{3^i}$ where $t_i \in \{0,2\}$.

However, the set of points whose ternary expansion uses only 0s and 2s is exactly the Cantor set from def. 5.0.1 (see rem. 5.0.6).

Therefore, the attractor of the IFS $\{F_0, F_1\}$ is $\{(x,0)^\top : x \in \text{Cantor set}\}$. □

9.4 Generated iterated function systems

We want to generalize our definition of IFSs and fractals so that we can play with things that look exactly like fractals (for example, where the scaling factor differs for each piece).

> **Definition 9.4.1** (affine transformation)
> A function $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $A \in M_n(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$. If $\mathbf{b} = \mathbf{0}$, we recover the linear transformations.
>
> We call F a linear contraction if there exists $0 < \lambda < 1$ such that $\|F(\mathbf{x}) - \mathbb{F}(\mathbf{y})\| < \lambda\|\mathbf{x} - \mathbf{y}\|$.

In general, "affine" just means linear but shifted.

Example 9.4.2. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $0 < \beta < 1$.

Then, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(\mathbf{x}) = (\beta A\mathbf{x} + \mathbf{b})$ is a linear contraction.

This linear contraction (1) scales by β, (2) rotates counter-clockwise by θ, and (3) translates by $\mathbf{b}$.

Definition 9.4.3 (compactness)
A subset $A \subseteq \mathbb{R}^n$ is compact if A is closed and bounded.

Write $\mathcal{K}_n$ for the set of all non-empty compact subsets of $\mathbb{R}^n$.

Definition 9.4.4 (generalized iterated function system)
Let $F_1, \dots, F_k : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be linear contractions. We call $F : \mathcal{K}_n \to \mathcal{K}_n$ given by

$$F(A) = F_1(A) \cup F_2(A) \cup F_3(A) \cup \cdots \cup F_k(A)$$

a (generalized) iterated function system.

This is well-defined since finite unions and the F_i's continuity preserve closure and compactness.

We will now:

1. Equip $\mathcal{K}_n$ with a metric.
2. Show F has a unique fixed point A^* and for all $A \in \mathcal{K}_n$, $F^n(A) \to A^*$. The point A^* is the attractor of F (and is a fractal!).

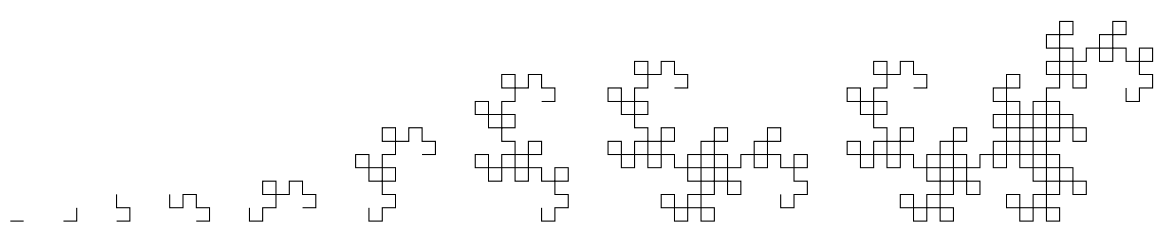
Example 9.4.5. Let $F_1(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x}$ and $F_2(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Find the attractor.

Solution. Notice that F_1 will (1) scale by $\frac{1}{\sqrt{2}}$ and (2) rotate by $\frac{\pi}{4}$. Then, F_2 will (1) scale by $\frac{1}{\sqrt{2}}$, (2) rotate by $\frac{3\pi}{4}$, and (3) shift one unit left.

Consider the line L from $(0, 0)$ to $(1, 0)$.

Then, we can draw:



This fractal, the dragon fractal tiles the space. □

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Mar 4

Remark 9.4.6. For all $A \in \mathcal{K}_n$, $F_i(A) \in \mathcal{K}_n$. This is because the continuous image of a compact set is compact (beyond the scope of this course).

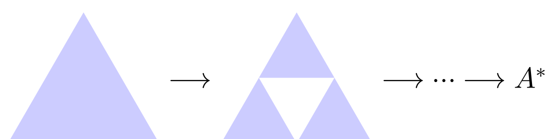
We can now equip $\mathcal{K}_n$ with a metric. We will consider $F : \mathcal{K}_n \to \mathcal{K}_n : A \mapsto F_1(A) \cup \cdots \cup F_k(A)$. This is well-defined since we already showed that the finite union of closed sets are closed, and it is trivial to show that the finite union of bounded sets is bounded.

We will then show that F has a unique fixed point $A^* \in \mathcal{K}_n$ and that for all $A \in \mathcal{K}_n$, $F^n(A) \to A^*$.

Example 9.4.7. Let $F_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}$, $F_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, and $F_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$.

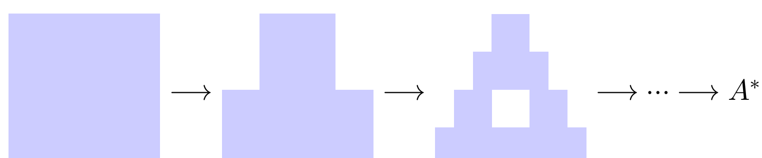
Find the attractor.

Solution. Let A be the filled triangle with vertices $(0,0)^\top$, $(1,0)^\top$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})^\top$:



This is the Sierpinski triangle.

Alternatively, we could have started with a square:



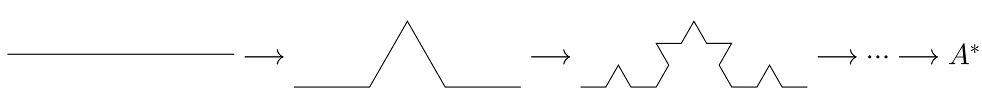
or with a goose:

TODO

but these all converge to the same attractor. □

Example 9.4.8. Repeat with $F_1(\mathbf{x}) = \frac{1}{3}\mathbf{x}$, $F_2(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$, $F_3(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{bmatrix}$, and $F_4(\mathbf{x}) = \frac{1}{3}\mathbf{x} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}$.

Solution. Let L be the line segment from $(0,0)^\top$ to $(1,0)^\top$. Then:



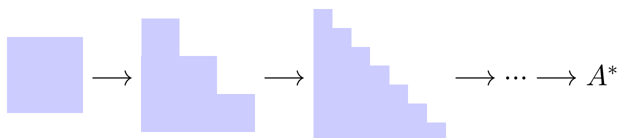
The attractor converges to the Koch curve. □

Example 9.4.9. Let $A = [0,1] \times [0,1]$ (i.e., the filled square).

Repeat with $F_1(\mathbf{x}) = \mathbf{x}$, $F_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $F_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Remark 9.4.10. Since F_1 is not a linear contraction, $\lim_{n\to\infty} F^n(A)$ will depend on A.

Solution. Draw the $[0,1] \times [0,1]$ square and iterate:



This is not a fractal by our strict definition (it is not even self-similar), but in our eyes and our hearts it's a fractal. □

Definition 9.4.11 (Hausdorff metric)
Let $\mathbf{v} \in \mathbb{R}^n$, $A, B \in \mathcal{K}_n$. First, define

$$d(\mathbf{v}, B) := \min\{\|\mathbf{v} - \mathbf{b}\| : \mathbf{b} \in B\}$$

(this should be an $\inf\{\cdots\}$ but since B is compact, the extreme value theorem gives us $\min\{\cdots\}$ instead)

Then, define

$$d(A, B) := \max\{d(\mathbf{a}, B) : \mathbf{a} \in A\}$$

i.e., the length of the longest direct path between points in A and B.

Finally, define

$$D(A, B) := \max\{d(A, B), d(B, A)\}$$

to fix the fact that d is not symmetric.

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Mar 6

Fact 9.4.12. D is a metric on $\mathcal{K}_n$

We take this fact without proof.

Example 9.4.13. $A = \{(1,1)\}$, let $B = \{(x, 0) : 0 \leq x \leq 1\}$

[figure]

> Then, $d(A, B) = 1$, $d(B, A) = \sqrt{2}$, and $D(A, B) = \max\{1, \sqrt{2}\} = \sqrt{2}$.

Lemma 9.4.14

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear contraction such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq \lambda\|\mathbf{x} - \mathbf{y}\|$ for some $\lambda \in (0, 1)$.

Then, for $A, B \in \mathcal{K}_n$, $D(f(A), f(B)) \leq \lambda D(A, B)$.

Proof. First, we have

$$d(f(a), f(B)) = \min_{b \in B} \|f(a) - f(b)\| \leq \min_{b \in B} \lambda\|a - b\| = \lambda \min_{b \in B} \|a - b\| = \lambda d(a, B)$$

and so

$$d(f(A), f(B)) = \max_{a \in A} d(f(a), f(B)) \leq \lambda \max_{a \in A} d(a, B) = \lambda d(A, B) \leq \lambda D(A, B)$$

Therefore, $d(f(A), f(B)) \leq \lambda D(A, B)$. Similarly, $d(f(B), f(A)) \leq \lambda D(A, B)$.

Hence, $D(f(A), f(B)) \leq \lambda D(A, B)$. □

Lemma 9.4.15

For $A_1, A_2, B_1, B_2 \in \mathcal{K}_n$,

$$D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$$

Proof. First,

$$\begin{aligned} d(A_1 \cup A_2, B_1 \cup B_2) &= \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2) \\ &= \max\left\{\max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2)\right\} \\ &\leq \max\left\{\max_{a \in A_1} d(a, B_1), \max_{a \in A_2} d(a, B_2)\right\} \qquad (\star) \end{aligned}$$

by the min in the definition.

$= \max\{d(A_1, B_1), d(A_2, B_2)\} \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$

Hence, $d(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$

Similarly, $d(B_1 \cup B_2, A_1 \cup A_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$

Therefore $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$ □

Lemma 9.4.16

Let $F_1, \cdots, F_k$ be linear contractions with contraction factor $\lambda \in (0, 1)$.

Consider $F : \mathcal{K}_n \to \mathcal{K}_n, F(A) = F_1(A) \cup F_2(A) \cup \cdots \cup F_k(A)$. Then, $D(F(A), F(B)) \leq \lambda D(A, B)$.

Proof. We have, $D(F(A), F(B)) \leq \max_{i=1,\dots,k} D(F_i(A), F_i(B))$ by lem. 9.4.15. By lem. 9.4.14, $\leq \max_{i=1,\dots,k} \lambda D(A, B) = \lambda D(A, B)$. □

Definition 9.4.17

Let (X, d) be metric space.

1. $(x_n) \subseteq X$ is Cauchy if $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$, such that $n, m \geq N \implies d(x_n, x_m) < \epsilon$.
2. X is complete if every Cauchy sequence $(x_n) \subseteq X$ converges to some $x \in X$.

Fact 9.4.18. (K_n, D) is complete.

We do not prove this.

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Mar 8

Theorem 9.4.19

Let $F_1, \dots, F_k$ be linear contractions with contraction factor $\lambda \in (0, 1)$.

Let $F : \mathcal{K}_n \to \mathcal{K}_n$ be the corresponding IFS. Then,

1. F has a unique fixed point A^*, which we call the attractor.
2. For all $A \in \mathcal{K}_n$, $F^m(A) \to A^*$.

Proof. Fix $A \in \mathcal{K}_n$. Consider its orbit $F^m(A)$. Look at the distance

$$D(F^{m+1}(A), F^m(A)) = D(F^m(F(A)), F^m(A)) \leq \lambda^m D(F(A), A)$$

by lem. 9.4.16. Let $\epsilon_m = \lambda^m D(F(A), A)$. Then, $\sum \epsilon_m$ converges, since $|\lambda| < 1$. Therefore, the sequence $(F^m(A)) \subseteq \mathcal{K}_n$ is strongly Cauchy. In particular, $F^m(A)$ is Cauchy, so there exists some $F^m(A) \to A^* \in \mathcal{K}_n$ because $\mathcal{K}_n$ is complete.

Since F is continuous, $F^{m+1}(A) \to F(A^*)$. Hence, $F(A^*) = A^*$.

Now, consider uniqueness. Suppose A^* and B^* are fixed points for F. Then,

$$D(A^*, B^*) = D(F(A^*), F(B^*)) \leq \lambda D(A^*, B^*)$$

but $\lambda \in (0, 1)$. This forces $D(A^*, B^*) = 0$, so $A^* = B^*$. □

Chapter 10

Complex Functions

Definition 10.1.1 (complex derivative)

Let $f : \mathbb{C} \to \mathbb{C}$. Then,

1. For $z_0 \in \mathbb{C}$, we say that
$$\lim_{z \to z_0} f(z) = L \in \mathbb{C}$$
if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon$$
2. The derivative of $f(z)$ at z_0 is
$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
provided the limit exists.

In general, we will write $f(x)$ for a real-valued function and $f(z)$ for a complex-valued function. Then, analogous to real-valued functions, we can consider complex fixed points.

Definition 10.1.2 (complex fixed points)

Let $a \in \mathbb{C}$ be a fixed point of $f(z)$. Then,

1. a is attracting if $|f'(a)| > 1$,
2. a is repelling if $|f'(a)| < 1$, and
3. a is neutral if $|f'(a)| = 1$.

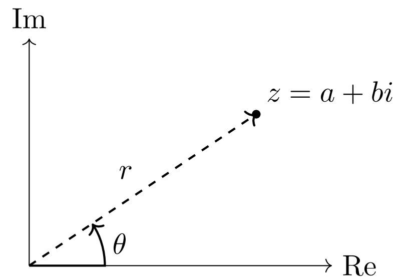
Remark 10.1.3 (attracting/repelling complex fixed point theorems)**.** We can obtain complex analogues of the proofs of the real-valued attracting/repelling fixed point theorems by replacing intervals around fixed points with open discs.

Example 10.1.4. Analyze the fixed points of $f(z) = z^2 + z + 1$.

Solution. The fixed points are $z^2 + z + 1 = z \iff z^2 + 1 = 0 \iff z = \pm i$.

Then, $f'(z) = 2z + 1$, so $|f'(i)| = |2i + 1| = \sqrt{5} > 1$ and $|f'(-i)| = |-2i + 1| = \sqrt{5} > 1$, so both are repelling. □

Recall polar form. For some complex number $z = a + ib$, we can plot it as (a, b):



Then, we can recall from MATH 135 that we can write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and we have really nice multiplication.

Fact 10.1.5 (PMℂ, MATH 135). $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$ and $(re^{i\theta})^n = r^n e^{in\theta}$, which is just so much prettier than Cartesian multiplication.

In particular, for complex numbers of the form $e^{2\pi i/n}$, we have $(e^{2\pi i/n})^n = e^{2\pi i} = 1$, which is a nice way to generate periodic points.

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Mar 11

Example 10.1.6. Let $z = e^{2\pi i/3}$ and $f(w) = w^2$.

Solution. Write $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Then, $f(z) = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $f^2(z) = e^{8\pi i/3} = e^{2\pi i/3} = z$.

That is, z is periodic with period 2.

We can then find $\left|(f^2)'(z)\right| = |f'(z)f'(f(z))| = \left|-1 + i\sqrt{3}\right| \cdot \left|-1 - i\sqrt{3}\right| = 4 > 1$, so z is attracting. □

Chapter 11

Julia Sets

11.1 Definition

Notation (quadratic family)**.** For $c \in \mathbb{C}$, write $Q_c(z) = z^2 + c$ just like the real one.

Definition 11.1.1

The filled Julia set for c is $K_c = \{z \in \mathbb{C} : (Q_c^n(z)) \text{ is bounded}\}$.

Equivalently, $\{z \in \mathbb{C} : \exists M > 0, \forall n \in \mathbb{N}, |Q_c^n(z) \leq M|\}$.

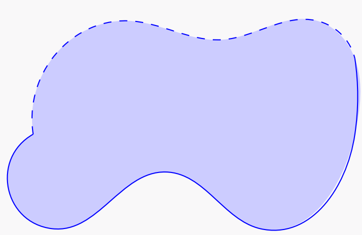
Remark 11.1.2. This is the complex analogue of Λ for $Q_c(x) = x^2 + c$ where $c \in \mathbb{R}$ and $c < -2$.

Definition 11.1.3

Let (X, d) be a metric space and $A \subseteq X$.

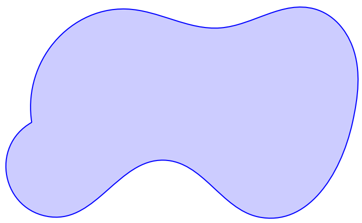
1. The closure of A is $\overline{A} = \{x \in X : \exists (a_n) \subseteq A, a_n \to x\}$.
2. The interior of A is $\text{Int}(A) = \{x \in X : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq A\}$.
3. The boundary of A is $\partial(A) = \overline{A} \setminus \text{Int}(A)$.

Example 11.1.4. Let A be the blob



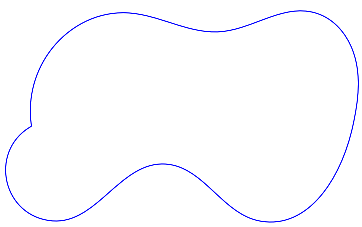
> Find the closure, interior, and boundary.

Solution. Since we can make a sequence of points that reaches the dashed open parts, the closure $\overline{A}$ will simply be



Then, since we can draw a ball on the shaded inside but not on the edge, the interior $\text{Int}(A)$ is

Finally, the boundary $\partial(A)$ is



□

> **Remark 11.1.5.** A is closed if and only if $A = \overline{A}$.

Lemma 11.1.6 (Assignment 4)

K_c is closed.

Definition 11.1.7

The Julia set for c is $J_c = \partial(K_c)$.

Remark 11.1.8. Since K_c is closed, $J_c = \partial(K_c) = \overline{K_c} \setminus \text{Int}(K_c) = K_c \setminus \text{Int}(K_c)$.

11.2 Construction

Example 11.2.1. Let $c = 0$, so $Q_0(z) = z^2$. What do K_0 and J_0 look like?

Solution. Let $z = re^{i\theta}$. Then, $|Q_0(z)| = |r^2 e^{2i\theta}| = r^2$. Likewise, $|Q_0^2(z)| = |r^4 e^{2i\theta}| = r^8$. Clearly, $|Q_0^2(z)| = r^{2^n}$. Therefore, $K_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ since that is when $|z|^{2^n}$ is bounded.

This is the unit disc in the complex plane. Therefore, $J_0 = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. □

Example 11.2.2. Repeat with $c = -2$.

Solution. First, let $R = \{z \in \mathbb{C} : |z| > 1\}$ and define a function $H : R \to \mathbb{C} : z \mapsto z + \frac{1}{z}$.

Then, we claim that H is injective. Suppose $H(z) = H(w)$. Then,

$$\begin{aligned} z + \frac{1}{z} &= w + \frac{1}{w} \\ zw &= z^2 + 1 - \frac{z}{w} \\ &= w^2 + 1 - \frac{w}{z} \\ w^2 - z^2 &= \frac{w}{z} - \frac{z}{w} = \frac{w^2 - z^2}{zw} \end{aligned}$$

This means that either $zw = 1$ or $w^2 - z^2 = 0$. However, $|zw| = |z| \cdot |w| > 1$, so $w = \pm z$. Since $H(w) = H(z)$, we must pick $w = +z$, and we are done.

Now, claim that $H : R \to \mathbb{C} \setminus [-2, 2]$ is surjective. Suppose that $H(z) = w$. Then,

$$\begin{aligned} z + \frac{1}{z} &= w \\ z^2 - wz + 1 &= 0 \\ z &= \frac{1}{2}(w \pm \sqrt{w^2 - 4}) \end{aligned}$$

and write z_+ or z_- for the two possible z's. Since these are roots of a polynomial with constant 1, we must have $z_+ z_- = 1$.

That is, either (1) $|z_+| > 1$ and $|z_-| < 1$, (2) $|z_+| < 1$ and $|z_-| > 1$, or (3) $|z_+| = |z_-| = 1$.

If either root is in R, then either $H(z_+) = w$ or $H(z_-) = w$.

Otherwise, $|z_+| = |z_-| = 1$. Then, $H(z) = H(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in [-2, 2]$.

Therefore, H is well-behaved (i.e., invertible) on $R \to \mathbb{C} \setminus [-2, 2]$.

Consider now $H(Q_0(z)) = H(z^2) = z^2 + \frac{1}{z^2}$. Note that $Q_{-2}(H(z)) = (z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z^2}$. Hence, $H(Q_0^n(z)) = Q_{-2}^n(H(z))$. *Lecture 25 Mar 13*

This looks quite similar to $S(Q_c^n(x)) = \sigma^n(S(x))$ in $\mathbb{R}$. We can say that H plays a similar role as S. In fact, (not course content), Q_0 and Q_{-2} are conjugate because H is a homeomorphism between them.

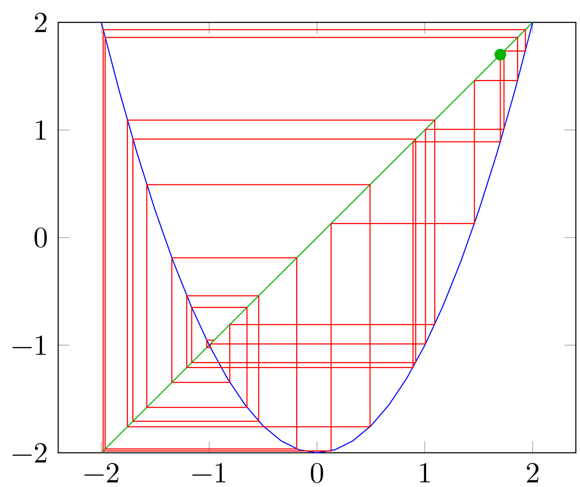
Let z_n be a diverging sequence $|z_n| \to \infty$. Note that $|H(z_n)| = \left|z_n + \frac{1}{z_n}\right| \geq |z_n| - \frac{1}{|z_n|} \to \infty$ Therefore, the image of the sequence $|H(z_n)| \to \infty$ also diverges.

Let $z \in \mathbb{C} \setminus [-2, 2]$. Since H is surjective, we know there exists a $w \in R$ such that $z = H(w)$, and see that

$$|Q_{-2}^n(z)| = |Q_{-2}^n(H(w))| = |H\underbrace{(Q_0^n(w))}_{\to\infty}| \to \infty$$

by the previous claim. Hence, $z \notin K_{-2}$ and we have that $K_{-2} \subseteq [-2, 2]$.

Finally, let $z \in [-2, 2]$. By graphical analysis,



there is no way to escape the box. That is, $z \in K_{-2}$, i.e., $[-2, 2] \subseteq K_{-2}$.

Therefore, $K_{-2} = [-2, 2]$, and we have that $J_{-2} = [-2, 2]$. ☐

Proposition 11.2.3 (Escape Criterion)
If $|z| \geq |c| > 2$, then $|Q_c^n(z)| \to \infty$. In particular, $z \notin K_c$.

Proof. We can write

$$|Q_c(z)| = \left|z^2 + c\right| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1)$$

Suppose $|z| > 2 + \lambda$ for some $\lambda > 0$. Then, we have that $|z| - 1 > 1 + \lambda$. Therefore, $|Q_c(z)| \geq |z|(1 + \lambda)$.

Iterating, we see that $|Q_c^n(z)| \geq |z|(1 + \lambda)^n \to \infty$. ☐

Corollary 11.2.4. Suppose $|c| > 2$. Then, $|Q_c^n(0)| \to \infty$ and $0 \notin K_c$.

Proof. Let $z = Q_c(0) = c$ and $|z| = |c| > 2$. By the Escape Criterion, $|Q_c^n(0)| \to \infty$. ☐

Corollary 11.2.5. Let $M = \max\{|c|, 2\}$. If $|z| > M$, then $|Q_c^n(z)| \to \infty$. That is, we have that $K_c \subseteq \{z : |z| \leq M\}$.

Proof. We have $|Q_c^n(z)| \geq (1+\lambda)^n |z| \to \infty$ by the proof of the Escape Criterion (not the Escape Criterion itself because we don't know if $|z| < 2$). □

Remark 11.2.6 (assignment hint!)**.** The fact that K_c is inside this bounded disc will help with the proof of its closedness.

Corollary 11.2.7. If there exists a k such that $\left|Q_c^k(z)\right| > \max\{|c|, 2\}$, then $|Q_c^n(z)| \to \infty$. That is, $z \notin K_c$.

Based on these results, we can develop the

Algorithm 1 Filled Julia set algorithm

```
Choose a large N ∈ ℕ.
for points z do
    if |Q_c^i(z)| > max{|c|, 2} for any i ≤ N then
        Colour z white
    else if |Q_c^i(z)| ≤ max{|c|, 2} for all i ≤ N then
        Colour z black
```

whose black-shaded region approximates K_c.

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Example 11.2.8. Is $i \in K_{2+i}$?

Solution. Let $Q(z) = z^2 + 2 + i$ and $M = \max\{\sqrt{5}, 2\} = \sqrt{5}$.

Then, $i \mapsto 1 + i \mapsto 2 + 3i$ but $|2 + 3i| = \sqrt{13} > \sqrt{5}$.

Therefore, $i \notin K_{2+i}$. □

Remark 11.2.9. For $n \in \mathbb{Z}$, $n \neq 0$, $\int_0^{2\pi} e^{int}\,\mathrm{d}t = 0$

Proof. Evaluate the integral:

$$\begin{aligned}
\int_0^{2\pi} e^{int}\,\mathrm{d}t &= \int_0^{2\pi} \cos(nt) + i\sin(nt)\,\mathrm{d}t \\
&= \int_0^{2\pi} \cos(nt) + i\int_0^{2\pi} \sin(nt)\,\mathrm{d}t \\
&= \left[\frac{1}{n}\sin(nt)\right|_0^{2\pi} + i\left[\frac{1}{n}\cos(nt)\right|_0^{2\pi} \\
&= 0
\end{aligned}$$

as desired. ☐

Proposition 11.2.10 (Cauchy's Estimate)

Let $P(z) = \sum_{n=0}^{d} a_n z^n$ be a polynomial such that $|P(z)| \leq M$ for all $|z - z_0| \leq r$.

Then, $|P'(z_0)| \leq \frac{M}{r}$.

Proof. Suppose $z_0 = 0$. Assume $|P(z)| \leq M$ for all $|z| \leq r$. Consider the integral

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{P(re^{it})}{re^{it}} \, \mathrm{d}t &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{d} a_n r^{n-1} e^{i(n-1)t} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} a_1 r^0 e^0 \, \mathrm{d}t && \text{(by rem. 11.2.9)} \\ &= a_1 \\ &= P'(0) \end{aligned}$$

and so we have

$$\begin{aligned} |P'(0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(re^{it})}{re^{it}} \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{r} \, \mathrm{d}t \\ &= \frac{M}{r} \end{aligned}$$

because $\left|re^{it}\right| = r \leq r$, so $\left|P(re^{it})\right| \leq M$.

Suppose now that $z_0 \neq 0$. Assume $|P(z)| \leq M$ for $|z - z_0| \leq r$. We proceed by just translating to make use of the first case.

Let $w = z - z_0$ so that $|P(w + z_0)| \leq M$ for all $|w| \leq r$. Then, by the first case,

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}w} \bigg|_{w=0} P(w + z_0) \right| &\leq \frac{M}{r} \\ \left| \frac{\mathrm{d}}{\mathrm{d}z} \bigg|_{z=z_0} P(z) \right| &\leq \frac{M}{r} \end{aligned}$$

completing the proof. ☐

Theorem 11.2.11

If z_0 is a repelling periodic point for $Q_c(z)$, then $z_0 \in J_c$.

Proof. Assume z_0 is a repelling periodic point with period n. Suppose for a contradiction that $z_0 \notin J_c$. Since z_0 is periodic, $z_0 \in K_c$. Therefore, z_0 is in the interior of the Julia set.

That is, $\exists r > 0$ such that $z \in K_c$ for all $|z - z_0| \leq r$ (i.e., there is an r-ball in K_c at z). For all z with $|z - z_0| \leq r$ and $K \in \mathbb{N}$, we have $\left|(Q_c^n)^k(z)\right| \leq M$ where $M = \max\{|c|, 2\}$ by the Escape Criterion.

Then, by Cauchy's Estimate, $\left|(Q_c^{nk})'(z_0)\right| \leq \frac{M}{r}$ for all $k \in \mathbb{N}$. Suppose that $\left|(Q_c^{nk})'(z_0)\right| = \lambda > 1$.

Finally, $\left|(Q_c^{nk})'(z_0)\right| = \prod_{j=0}^{k-1} \left|(Q_c^{n})'(Q_c^{j}(z_0))\right| = \lambda^k \to \infty \not\leq \frac{M}{r}$, by prop. 3.2.7, which is our conradiction. Therefore, $z_0 \in J_c$. □

Fact 11.2.12. Suppose $K \subseteq \mathbb{C}$ is closed. Then, if $z \in \text{Int}(K)$, then $Q_c(z) \in \text{Int}(Q_c(K))$.

Proposition 11.2.13

If $Q_c(z) \in J_c$, then $z \in J_c$. That is, the Julia set is closed under preimages.

Proof. If $Q_c(z) \in J_c$, then $Q_c(z) \in K_c$ and $z \in K_c$.

But if $Q_c(z) \notin \text{Int}(K_c)$, then $z \notin \text{Int}(K_c)$.

Therefore, $Q_c(K_c) = K_c$. □

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Definition 11.2.14 (supersensitivity)

We say Q_c is supersensitive at z_0 if whenever $z_0 \in U \subseteq \mathbb{C}$ is open, then $\mathbb{C} = \cup_{n=0}^{\infty} Q_c^n(U)$.

Fact 11.2.15. Q_c is supersensitive at all $z_0 \in J_c$.

For a geometric justification, see https://agony.retrocraft.ca/PMATH370/doodles#fact-11215.

Therefore, if we pick $z \in \mathbb{C}$ and $z_0 \in J_c$, then for all $\varepsilon > 0$, $U = B_\varepsilon(z_0)$, by supersensitivity, $Q_c^k(w) = z$ for some $w \in U$. By looking at the backwards orbit of z, we can find a very close $w \in \mathbb{C}$ to J_c. This leads to a new algorithm:

Algorithm 2 Algorithm to draw the Julia set

1: Choose $z \in \mathbb{C}$.
2: Compute 10,000 terms in the backwards orbit, randomly selecting a preimage at each step.
3: Plot all but the first 100 points.

This is implemented here: https://marksmath.org/visualization/julia2.html

Chapter 12

The Mandlebrot Set

12.1 Construction

Definition 12.1.1

Let (X, d) be a metric space. A path from a to b is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

A set $A \subseteq X$ is path-connected if for all $a, b \in A$, there exists a path $\gamma : [0, 1] \to X$ from a to b such that $\gamma([0, 1]) \subseteq A$.

The maximal path-connected subsets of A are the path-connected components of A.

If the path-connected components are all singletons, then A is totally disconnected.

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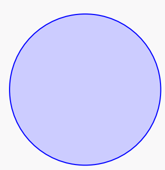
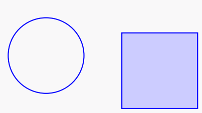
Example 12.1.2. The unit circle $A = \{z : |z| \leq 1\}$:



is path-connected.

Example 12.1.3. The set A:



is neither path-connected nor totally disconnected.

Example 12.1.4. The Cantor set $K \subseteq \mathbb{R}$ is totally disconnected.

Theorem 12.1.5 (all-or-nothing theorem)
For $Q_c(z) = z^2 + c$, either

1. $|Q_c^n(0)|$ is bounded (i.e., $0 \in K_c$), in which case K_c is path-connected; or
2. $|Q_c^n(0)| \to \infty$ (i.e., $0 \notin K_c$), in which case K_c is totally disconnected.

Definition 12.1.6 (Mandelbrot set)
The set $\mathcal{M} = \{c \in \mathbb{C} : |Q_c^n(0)| \text{ is bounded}\} = \{c \in \mathbb{C} : K_c \text{ is path connected}\}$.

12.2 Shape

We want to prove that the Mandlebrot set has our expected shape of circle + heart.

Recall from cor. 11.2.4 of the Escape Criterion that if $|c| > 2$, then $|Q_c^n(0)| \to \infty$.

Example 12.2.1. We know $0 \in \mathcal{M}$, so K_0 (the disc, ex. 11.2.1) is path-connected. Likewise, $-2 \in \mathcal{M}$, so K_{-2} (the interval, ex. 11.2.2) is path-connected.

Example 12.2.2. Let $c = 2$. Then, $Q_c(z) = z^2 + 2$ and $0 \mapsto 2 \mapsto 6 \mapsto 38 \mapsto \cdots \mapsto \infty$. That is, $2 \notin \mathcal{M}$.

Example 12.2.3. Let $c = i$. Then, $Q_c(z) = z^2 + i$ and $0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto 1 \mapsto \cdots$, which means that $i \in \mathcal{M}$.

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Remark 12.2.4. Pick a rational $z \in Q_c$. Let $M = \max\{|c|, 2\}$. By the Escape Criterion, either

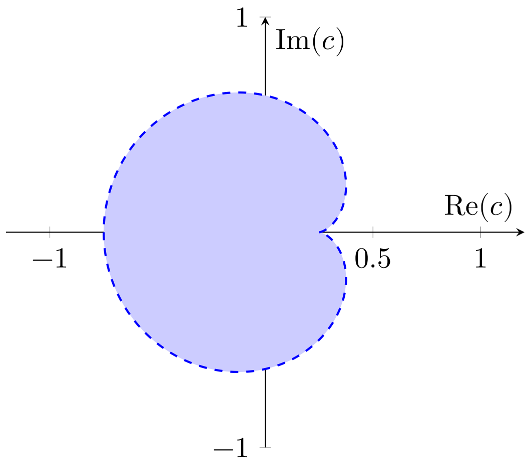
1. $|Q_c^n(z)| \leq M$ for all n, or
2. $|Q_c^n(z)| \to \infty$.

First, we want to determine when $Q_c(z)$ has an attracting fixed point. Why do we care? Suppose $z \in \mathbb{C}$ is an attracting fixed point for Q_c. Then, there exists $r > 0$ such that $x \in B_r(z)$, giving $Q_c^n(x) \to z$ and $B_r(z) \subseteq K_c$. Hence, K_c is path connected and $c \in \mathcal{M}$. That is, the existence of an attracting fixed point for Q_c tells you that $c \in \mathcal{M}$.

Suppose $z \in \mathbb{C}$ exists. Then, $z^2 + c = z$ and $|2z| < 1$. This implies $c = z - z^2$ and $|z| < \frac{1}{2}$. We can parametrize the boundary. Write $z = \frac{1}{2}e^{i\theta}$ in polar form, so that

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{i2\theta}, \quad \theta \in \mathbb{R}$$

which, when plotted for all values of θ, gives the cartioid:



Second, when does Q_c have an attracting 2-cycle? That is, when does Q_c admit a periodic point of period 2? If Q_c has an attracting periodic point z with period 2, we can similarly show that $c \in \mathcal{M}$ (do the open ball around the two cycle, and the existence of that ball makes K_c path-connected).

Suppose $(z^2 + c)^2 + c = z$. Then, $p(z) := z^4 + 2cz^2 - z + c^2 + c = 0$. The roots of $p(z)$ include all the points with period ≤ 2: both the 2-cycle points and the fixed points.

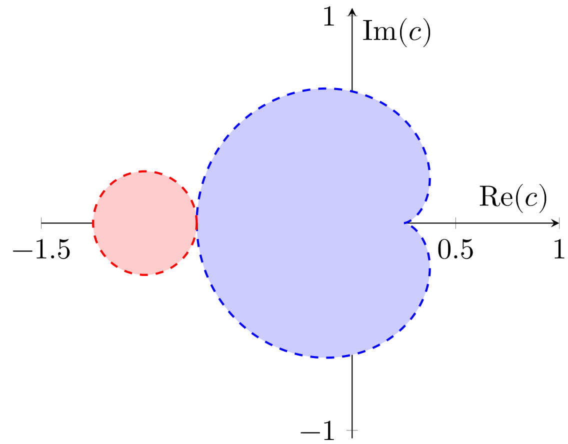
Let p_1 and p_2 be the fixed points of Q_c. Then, $(z - p_1)(z - p_2) = z^2 - z + c$ is a factor of $p(z)$.

That is, z is a root of $\frac{z^4+2cz^2-z+c^2+c}{z^2-z+c} = z^2 + z + c + 1$.

Let z_1 and z_2 be the roots of $z^2 + z + c + 1$, i.e., the period-2 points for Q_c. For these to be attracting, we must have

$$\begin{aligned} |(Q_c^2)'(z_i)| < 1 &\implies |Q_c'(z_1) - Q_c'(z_2)| < 1 \\ &\implies 4|z_1 z_2| < 1 \\ &\implies |z_1 z_2| < \frac{1}{4} \\ &\implies |c + 1| < \frac{1}{4} \end{aligned}$$

which is a ball of radius $\frac{1}{4}$ centered at -1:



Proposition 12.2.5

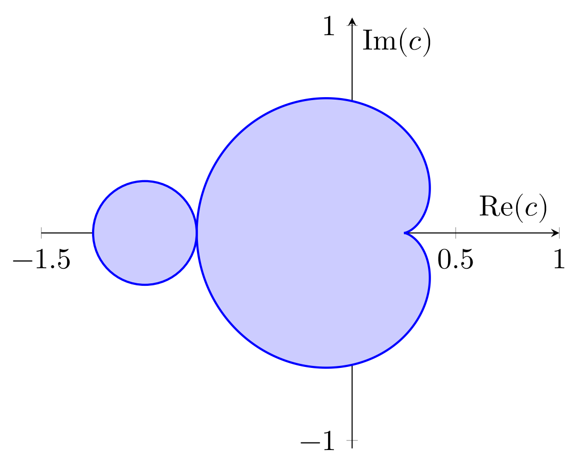
We can write $\mathcal{M}$ as the intersection

$$\{c \in \mathbb{C} : |c| \leq 2\} \cap \{c : |c^2 + c| \leq 2\} \cap \{c : |(c^2 + c)^2 + c| \leq 2\} \cap \cdots$$

for repeated iterations of Q_c^n.

Proposition 12.2.6

$\mathcal{M}$ is closed.



Chapter 13

Polynomial Julia Sets

Lecture 30
Mar 25

Theorem 13.1.1 (Polynomial Escape Criterion)
Let $p(z) = a_n z^n + \dots + a_1 z + a_0$ be a complex polynomial with $a_n \neq 0$ and $n \geq 2$. Then, there exists $R > 0$ depending only on n and a_i such that $\left|p^k(z)\right| \to \infty$ for all $|z| \geq R$.

Informally, there is a ball of radius z outside of which iteration blows up.

Proof. Fix $\lambda > 1$ and let $C = \sum_{i=0}^{n-1} |a_i|$.

Consider $R = \max\left\{1, \frac{2c}{|a_n|}, \left(\frac{2\lambda}{|a_n|}\right)^{1/n-1}\right\}$.

Assume $|z| \geq R$. Then,

$$\begin{aligned}
|p(z)| &\geq |a_n z^n| - \left|a_{n-1} z^{n-1} + \dots + a_1 z + a_0\right| \\
&\geq |a_n z^n| - \left(|a_{n-1}| \cdot |z|^{n-1} + \dots + |a_1| \cdot |z| + |a_0|\right) \\
&\geq |a_n z^n| - C|z|^{n-1} && (\text{since } |z| \geq R \geq 1) \\
&= |z|^n \left(|a_n| - \frac{C}{|z|}\right) \\
&\geq |z|^n \left(|a_n| - \frac{|a_n|}{2}\right) && \left(\text{since } |z| \geq \tfrac{2C}{|a_n|} \implies \tfrac{C}{|z|} \leq \tfrac{|a_n|}{2}\right) \\
&= |z| \cdot \frac{1}{2}|a_n| \cdot |z|^{n-1} \\
&\geq |z| \cdot \frac{1}{2}|a_n| \cdot \frac{2\lambda}{|a_n|} && \left(\text{since } |z| \geq \left(\tfrac{2\lambda}{|a_n|}\right)^{1/n-1} \implies |z|^{n-1} \geq \tfrac{2\lambda}{|a_n|}\right) \\
&\geq \lambda |z|
\end{aligned}$$

Therefore, $\left|p^k(z)\right| \geq \lambda^k |z|$ so it blows up to infinity. □

Remark 13.1.2. Either (1) for all k, $\left|p^k(z)\right| < R$, or (2) $\left|p^k(z)\right| \to \infty$.

Definition 13.1.3

Let $p(z)$ be a complex polynomial with degree ≥ 2.

The filled Julia set of $p(z)$ is $K_p = \{z \in \mathbb{C} : p^k(z) \text{ is bounded}\}$.

The Julia set of $p(z)$ is $J_p = \partial(K_p)$.

We can approximate the Julia set by iterating and seeing if a given point "escapes" in a fixed amount of time.

Chapter 14

Guest Lectures

14.1 Joaco Prandi: Creating a sundial

Lecture 31
Mar 27

Definition 14.1.1 (box-counting dimension)
Consider a blob K. Given a grid with spacing δ, define $N_\delta(K)$ to be the number of squares touched by K.

The box-counting dimension is $\dim_B(K) = \lim_{\delta \to 0} \frac{\log(N_\delta(K))}{-\log(\delta)}$.

This does not always exist, so we define the upper and lower box-counting dimensions as $\overline{\dim}_B(K) = \limsup_{\delta \to 0} \frac{\log(N_\delta(K))}{-\log(\delta)}$ and $\underline{\dim}_B(K) = \liminf_{\delta \to 0} \frac{\log(N_\delta(K))}{-\log(\delta)}$, which always exist.

The definition of $N_\delta(K)$ can be replaced by a lot of other vaguely similar ideas:

- the maximal number of δ-balls that pack into K
- the minimal number of δ-balls that cover K
- etc.

recovering an equivalent definition.

Definition 14.1.2 (Hausdorff dimensional measure)
Define $\mathcal{H}^s_\delta(K) = \inf\left\{\sum_{n=0}^{\infty} |U_n|^s : \bigcup_{n=0}^{\infty} U_n \supset K, |U_n| \leq \delta\right\}$ where the U_n's are a cover of K and $|U_n|$ is the diameter of the set U_n.

Then, let $\mathcal{H}^s(K) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(K)$.

When working in $\mathbb{R}^d$, we have that $\mathcal{H}^n$ for $n \leq d$ measures the n-dimensional Euclidean metric. That is, $\mathcal{H}^1$ measures length, $\mathcal{H}^2$ measures area, etc.

Fact 14.1.3. Let $s < t$. If $\mathcal{H}^s(K)$ is finite, then $\mathcal{H}^t(K) = 0$.

If $\mathcal{H}^t(K)$ is non-zero and finite, then $\mathcal{H}^s(K) = \infty$.

This means that the values will go $\{\dots, \infty, \infty, \text{some non-zero finite value}, 0, 0, \dots\}$.

Definition 14.1.4 (Hausdorff dimension)

The Hausdorff dimension $\dim_H(K) = \sup\{s : \mathcal{H}^s(K) = \infty\} = \inf\{s : \mathcal{H}^s(K) = 0\}$.

Example 14.1.5. If C is the Cantor set, then $\dim_B(C) = \dim_H(C) = \log_3(2)$.

If $F = \{\frac{1}{n}\}_{n=0}^{\infty}$, then $\dim_B(F) = \frac{1}{2}$ (for some reason) and $\dim_H(F) = 0$.

Fact 14.1.6. In general, the box-counting dimension is invariant under closures, while the Hausdorff dimension varies after closure.

Fact 14.1.7. If $\dim_H(F) < 1$, then the set F is totally disconnected.

Fact 14.1.8. If f is Lipschitz with ratio c, then $\mathcal{H}^s(f(K)) \le c^s \mathcal{H}^s(K)$.

This implies that $\dim_H(f(K)) \le \dim_H(K)$.

Let L_θ be the line through the origin with angle $\theta \in [0, \pi)$ and $\text{Proj}_\theta(F)$ be the orthogonal projection to L_θ.

Theorem 14.1.9

For almost all θ (that is, in all cases that actually matter),

1. If $\dim_H(F) \le 1$, then $\dim_H(\text{Proj}_\theta(F)) = \dim_H(F)$.
2. If $\dim_H(F) > 1$, then $\dim_H(\text{Proj}_\theta(F)) = 1$ and $\mathcal{H}^1(\text{Proj}_\theta(F)) > 0$.

Informally, everything is either less than one-dimensional or casts a one-dimensional shadow.

Given a set which casts a shadow on L_{θ_1} and L_{θ_2}, we can split it up and rotate the sections so that it the projection on L_{θ_1} becomes a set of singletons but the projection on L_{θ_2} remains connected (this is the iterated Venetian blinds process).

Theorem 14.1.10

Let $G_\theta \subset L_\theta$ for $\theta \in [0, \pi)$ be a collection of sets such that $\bigcup_\theta G_\theta$ is a measurable 2-dimensional set. Then, there exists a set $F \subseteq \mathbb{R}^2$ such that $G_\theta \subset \text{Proj}_\theta(F)$ and $\mathcal{H}^1(\text{Proj}_\theta(F) \setminus G_\theta) = 0$ for almost all θ.

14.2 Paul Fieguth: Bifurcations in continuous- and discrete-time systems

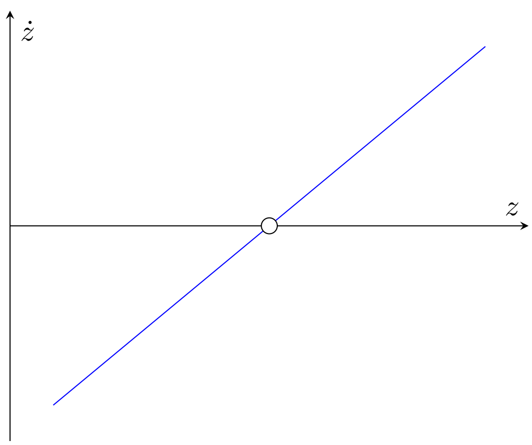
Lecture 32
Apr 1

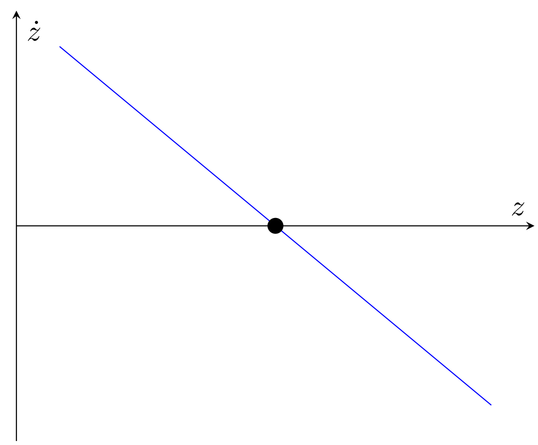
In general, university courses focus on “nice” linear, Gaussian, small models. However, reality is usually non-linear, non-Gaussian, and large. We will look at non-linear systems, in particular, bifurcations in non-linear dynamics.

Recall that in continuous time, we define some system as $\dot{z}(t) = f(z(t), \theta)$; in discrete time, we have $z_{n+1} = \bar{f}(z_n, \theta)$.

Discrete time can be expressed as the forward Euler discretization of continuous time, i.e., if $\dot{z} = f(z)$, we have $z(t + \delta) = z(t) + \delta \cdot f(z(t))$.

Suppose we draw a system diagram relating z to $\dot{z}$. When $f(z)$ crosses the z-axis, we have a fixed point because $\dot{z} = 0$.





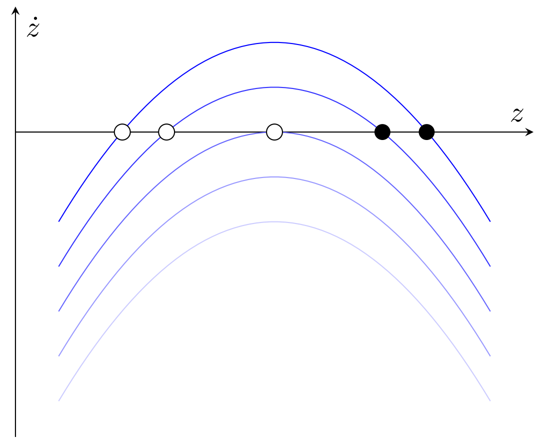
When sloping up the derivative is positive to the right and negative to the left. That means it is pushing away from the fixed point, creating instability. In the downwards-sloping figure, the fixed point is attracting (stable).

Linear systems have a handful of key attributes:

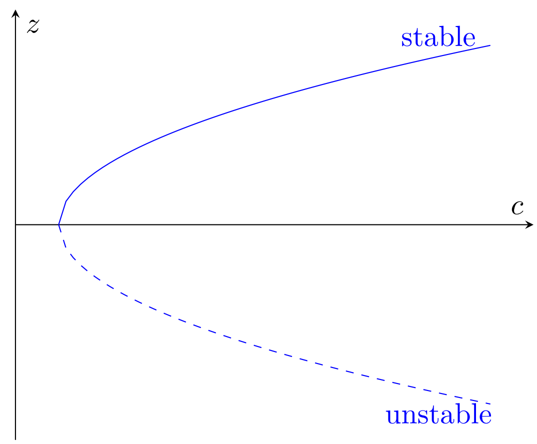
- Superposition: If $x_1 \mapsto y_1$ and $x_2 \mapsto y_2$, then $\alpha x_1 + \beta x_2 \mapsto \alpha y_1 + \beta y_2$
- Sine wave: If $A \sin(\omega t + \phi)$ goes in, then $B \sin(\omega t + \varphi)$ comes out. Only the phase and amplitude can be changed, not the frequency.
- Constant input cannot lead to oscillating output.

Definition 14.2.1 (bifurcation)
Discontinuous change in an attribute or behaviour in response to a continuous change in parameter.

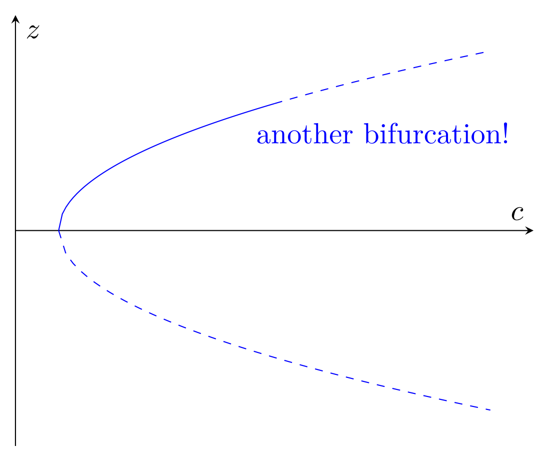
Suppose $\dot{z} = -(z - 5)^2 + c$:



Then, the number of fixed points jumps from none to two as c crosses some value. To summarize this, we can draw a bifurcation plot:

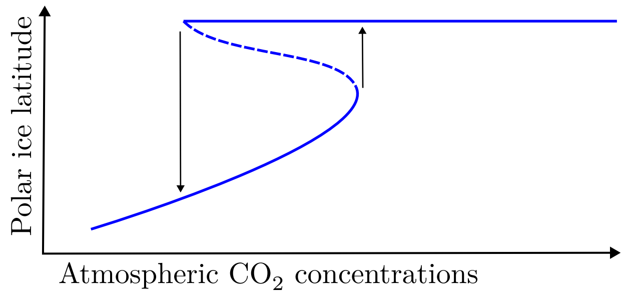


We also have bifurcations in discrete time, when we punch through the 45-degree line (as covered in the course proper). Considering the quadratic family as a discrete system, we can draw a bifucation plot



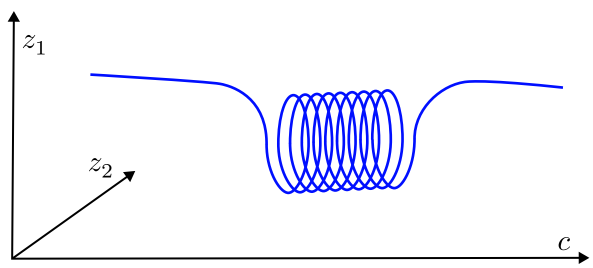
since the stable points eventually become unstable.

There are two major kinds of bifurcations that show up. First, the double-fold:

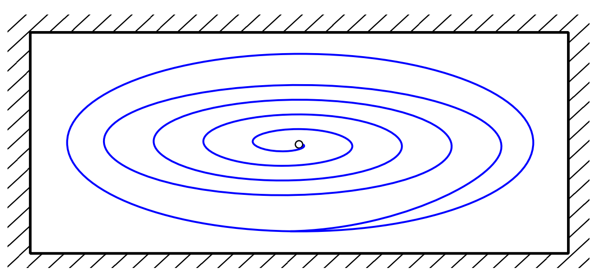


This is the way that thermostats, switches, etc. work to force the system into one of the two stable states. That is, global climate models are just fridges.

Second, the Hopf bifurcation: a transition between cycling and not cycling.



Cycling usually comes up from a bounded unstable system. Instability forces the system away from the fixed point, but the bounds prevent it from leaving, leading to a cyclical motion.



Ohter bifurcations include stick-slips (like chalkboard dashed lines), Covid cases, slap bracelets, jumping frogs, ecologies, epileptic seizures, etc.

14.3 Andy Zucker: Fixed point properties in topological dynamics

Lecture 33
Apr 3

Definition 14.3.1

Let G be a group. A G-flow is a compact Hausdorff space X equipped with a continuous action $a : G \times X \to X$ satisfying (1) $a(1_G, x) = x$ and (2) $a(g, a(h, x)) = a(gh, x)$.

Notation. Typically, we understand actions as implied and write $g.x$ or just gx for $a(g, x)$.

For example, the above axioms can be written as $1_G x = x$ and $g(hx) = (gh)x$.

Example 14.3.2. Let $G = \mathbb{Z}$ and X be the unit circle.

Let $T : X \to X$ be rotation by an irrational α, which acts as our generating homomorphism. Since α is irrational, the action is free, i.e., for all x and non-zero n, $T^n(x) \neq x$.

Example 14.3.3. Let $G = \mathbb{Z}$ and $X = 2^{\mathbb{Z}}$.

Let $T : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the Bernoulli shift $T(x)(n) = x(n-1)$.

This action is not free, since $x \equiv 0$ is a fixed point of T. However, there is a closed, non-empty, T-invariant subspace of X which is free.

These two examples are related. Suppose we define an interval on the unit circle. Then, check if each of the elements of the orbits of the rotation fall in that interval, and assign binary values to the function based on that. This generates the closed, non-empty, T-invariant subspace.

For now, fix X as the Cantor space Σ. Write $\mathrm{Clop}(X) = \{A \subseteq X : A \text{ is both closed and open}\}$.

Definition 14.3.4

A probability measure on X is a map $\mu : \mathrm{Clop}(X) \to [0, 1]$ with $\mu(X) = 1$ and finite additivity.

Example 14.3.5. View X as $2^{\mathbb{Z}}$. For each $A \in \mathrm{Clop}(X)$ defined by $\{x : x(n) = i\}$ for some fixed n, set $\mu(A) = \frac{1}{2}$. Generate the rest of the values axiomatically.

This is like flipping the n^{th} coin and expecting i.

Example 14.3.6. Fix $x \in X$. The Dirac delta at x is the measure $\mu(A) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$

This is measuring whether a set contains x.

Definition 14.3.7

The space of probability measures $P(X)$ is equipped with a topology such that $\mu_n \to \mu$ if and only if for all clopen A, $\mu_n(A) \to \mu(A)$.

This is the "weak* topology" on $P(X)$.

If G is a countable group and G acts on X, then G acts on $P(X)$ which we define as $(g.\mu)(A) = \mu(g^{-1}.A)$.

Definition 14.3.8

A countable group is amenable if for any finite $S \subseteq G$ and $\varepsilon > 0$, there exists an $F_{S,\varepsilon}$ such that $\frac{|SF \setminus F|}{|F|} < \varepsilon$.

Example 14.3.9. The integers $\mathbb{Z}$ are amenable.

Consider $S = \{\pm 1\}$. Given $\varepsilon > 0$, find $n \in \mathbb{N}$ such that $\frac{2}{n} < \varepsilon$. Let $F_{S,\varepsilon}$ be an interval of length n.

Then, $|SF \setminus F| = 2$ giving $\frac{|SF \setminus F|}{|F|} < \varepsilon$.

Theorem 14.3.10 (Følner)

A countable group G is amenable if and only if whenever G acts on $X = \Sigma$, the induced action on $P(X)$ has a fixed point.

Proof. Suppose G is amenable. Write G as an increasing union $\bigcup S_n$ where $S_1 \subseteq \S_2 \subseteq \cdots$ are finite. Let $F_n \subseteq G$ be $(S_n, \frac{1}{n})$-Følner. Given an action of G on the Cantor space X,

1. Pick an arbitrary $x \in X$.
2. For every $n \in \mathbb{N}$, let $\mu_n = \sum_{g \in F_n} \frac{1}{|S_n|} \delta_{gx}$ where δ is the Dirac delta. That is, we are taking the average of the Dirac measures of finitely many points. Concretely, given a clopen set A, we ask what proportion of the points lie inside of A.

Because $\langle F_n : n \in \mathbb{N} \rangle$ are more and more Følner, we have $\mu_n(A) - (g.\mu_n)(A) \to 0$ for every $A \in \mathrm{Clop}(X)$ and $g \in G$.

That is, because g eventually lies in one of the S_n and $\frac{1}{n}$ is very small, applying the action will eventually... something... i'm lost...

Then we use the compactness of $P(X)$ to do something? which passes to the convergent subsequence of μ_n with limit μ? those are math words! they mean something! □

List of Named Results

Index of Defined Terms